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Incremental updating approximations for double-quantitative decision-theoretic rough sets with the variation of objects[☆]

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ABSTRACT

Double-quantitative decision-theoretic rough sets (Dq-DTRS) provide more comprehensive description methods for rough approximations of concepts, which lay foundations for the development of attribute reduction and rule extraction of rough sets. Existing researches on concept approximations of Dq-DTRS pay more attention to the equivalence class of each object in approximating a concept, and calculate concept approximations from the whole data set in a batch. This makes the calculation of approximations time consuming in dynamic data sets. In this paper, we first analyze the variations of equivalence classes, decision classes, conditional probability, internal grade and external grade in dynamic data sets while objects vary sequentially or simultaneously over time. Then we propose the updating mechanisms for the concept approximations of two types of Dq-DTRS models from incremental perspective in dynamic decision information systems with the sequential and batch variations of objects. Meanwhile, we design incremental sequential insertion, sequential deletion, batch insertion, batch deletion algorithms for two Dq-DTRS models. Finally, we present experimental comparisons showing the feasibility and efficiency of the proposed incremental approaches in calculating approximations and the stability of the incremental updating algorithms from the perspective of the runtime under different inserting and deleting ratios and parameter values.

1. Introduction

Rough set theory is an effective mathematical tool for studying inaccurate and uncertain knowledge from information systems [1], which has been widely used in decision supporting, cloud computing, machine learning and intelligent information processing. In the contemporary era, uncertainties of data have increased dramatically due to the increase of collection means and the amount of data. As an effective uncertainty analysis tool, rough set theory attracts great attention. At present, some researchers are studying how to calculate concept approximations quickly, because it is the indispensable cornerstone of knowledge representation and feature selection in rough sets [2–4].

In classical Pawlak model, concept approximations mainly consider the inclusion and intersection relationships between an approximated concept and basic knowledge generated by conditional attributes in the universe. It is a qualitative model with no fault tolerance capability. Considering the technical limitations and other uncertainties in data collection, we find that some actual data may have missing values and small fluctuations. So this classical model has been popularized to make it fault-tolerant, thus enhancing its applicability. Different quantitative extension models of rough sets are proposed such as tolerance rough sets [5], graded rough sets [6], variable precision rough sets [7], probabilistic rough sets [8], decision-theoretic rough sets (DTRS) [9,10] and fuzzy variable precision rough sets [11]. Moreover, the development and application of DTRS are remarkable. Yang et al. [12] and [13] put forward sequential three-way methods for multi-class decision and dynamic hybrid data with the temporal–spatial composite variation, respectively. Fujita et al. [14] first put forward the proposal of applying three-way decisions to the resilience analysis of critical infrastructures. More noteworthy is that some composite models based on DTRS with double fault tolerance capabilities are proposed, which thoroughly describe the approximation space through the double quantitative indicators with complementary relationship.

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Li et al. [15] constructed Dq-DTRS based on graded and decision-theoretic rough set models. Yu et al. [16] established multi-granulation Dq-DTRS models. Xu et al. [17] proposed generalized multi-granulation Dq-DTRS models. Fan et al. [18] proposed logical double-quantitative rough fuzzy sets. Guo et al. [19] and [20] proposed logical double-quantitative rough fuzzy sets and local logical disjunction double-quantitative rough sets, respectively.

All of the above composite models study concept approximations under static decision information systems. However, the object set, attribute set and attribute values of decision information systems may vary individually or associatively over time in sequences or in batches. How to comprehensively and efficiently approximate concepts in dynamic decision information systems is our focus. Incremental learning is an efficient technology of dynamic data mining, which can acquire knowledge from current data more quickly based on the prior knowledge from previous data and the correlation of real time data. It can be used not only to process dynamic data such as dynamic multi-source data [21], but also to process large data from the perspective of updating data sequences [22,23]. Researchers study incremental updating of rough sets from different aspects such as incremental approximations [24], incremental decision rules [25] and incremental feature selection [26,27]. This paper focuses on incremental approximation updating in dynamic decision information systems.

When the object set varies over time, Chen et al. [28] studied systematically the approximation updating mechanisms of variable precision rough sets. Luo et al. [29] established the incremental approximation updating mechanism of DTRS based on matrix form. Liu et al. [30] studied the incremental updating approach of interesting knowledge. Cheng [31] proposed dynamic maintenance approaches of approximations for fuzzy rough sets. Li et al. [32] proposed an incremental method for fast updating approximations of dominance-based rough sets. Yu et al. [24] and Luo et al. [33] analyzed profoundly the incremental updating mechanisms of rough sets in interval-valued and set-valued systems, respectively. Liu et al. [34] and [35] proposed the incremental learning methods of knowledge discovery in incomplete information systems and business intelligent systems, respectively. When the attribute set varies over time, Chan et al. [36] first studied the incremental approximation updating of rough sets. Liu et al. [37] studied systematically incremental approximation updating of probabilistic rough sets. Some researchers have studied incremental approximation updating in set-valued ordered decision systems [38], hierarchical multi-criteria decision systems [39] and interval-valued ordered information system [40]. When the attribute values varies over time, Chen et al. [41] studied the incremental approximation updating mechanisms of rough sets. Zeng et al. [42] proposed dynamically updating approximation approaches of fuzzy rough sets when coarsening or refining the attribute values. Li et al. [43] and Hu et al. [44] put forward the incremental approaches for dynamic maintenance of approximations of dominance-based rough sets and multi-granulation rough sets, respectively. Luo et al. [45] studied the updating mechanisms of three-way decisions in incomplete multi-scale information systems with the variation of scales. In addition, some researchers study incremental approximations with the simultaneous variation of the object set, attribute set and attribute values. Wang et al. [46] presented the efficient incremental approximation updating methods in ordered information systems with the simultaneous variation of the object set and attribute values. Yang et al. [47] proposed a unified framework for the incremental updating of the probabilistic regions of DTRS with multilevel variations of the object set, attribute set and attribute values.

At present, there is little research on the dynamic maintenance of approximations of composite models. In view of the large scale, fast updating speed and the uncertainty of data, incremental approximation updating of composite models contribute to efficient expression of knowledge and rules. This paper studies the incremental approximation updating of a representative composite model in dynamic decision information systems. Considering that (1) Dynamic data sets with the variation of objects are ubiquitous, such as credit card applications, spam classification, TV dramas and movie recommendation systems etc.; (2) The computational complexity of rough approximations is positively correlated with the square of the number of objects, which makes the computation of approximations time-consuming or even infeasible in large-scale dynamic data sets; (3) The static calculation method of approximations takes up a large amount of storage and the calculation speed is slow, so it cannot satisfy the requirements of fast analysis and decision-making in the era of rapid data updating; our research background is the dynamic decision system with the variation of objects. Combining the reasonable semantic interpretation for decision-making process and the completeness of describing approximations of Dq-DTRS and the efficiency of incremental learning, we study the incremental approximation updating of Dq-DTRS in dynamic decision systems with the sequential and batch variations of objects, which can help to update quickly the approximations and promote feature selection and rule extraction of Dq-DTRS in dynamic and large-scale data sets.

The main contributions of this paper are as follows: (1) From the perspective of incremental learning, we make full use of prior knowledge to study dynamic approximation updating mechanisms of the composite model Dq-DTRS. (2) Efficient incremental sequential and batch algorithms for updating approximations of two Dq-DTRS models are proposed in dynamic decision systems with the insertion and deletion of objects in sequences or in batches. (3) Considering the influence of different inserting and deleting ratios and parameter values on the performance of incremental algorithms, we analyzed the stability of incremental algorithms. (4) The research of incremental approximations in this paper provides a basis for the incremental feature selection and rule extraction of composite models.

The rest of the paper is organized as follows. In Section 2, we mainly reviewed some basic knowledge of decision information systems and two Dq-DTRS models. In Section 3, we study systematically incremental approximation updating approaches of two Dq-DTRS models in dynamic decision systems with the sequential and batch insertions of objects, respectively. In Section 4, we explored deeply the incremental approximation updating approaches of Dq-DTRS models in dynamic decision systems with the sequential and batch deletions of objects, respectively. In Section 5, two static algorithms and eight incremental algorithms are proposed to verify the feasibility and efficiency of the incremental approximation updating approaches. In Section 6, experimental results show the computational efficiency and stability of our proposed incremental algorithms in calculating approximations. Section 7 concludes the paper and elaborates future studies.

2. Preliminaries

We first introduce some basic concepts about decision information systems and two types of Dq-DTRS models.

Let $S = (U, A, V, f)$ be a decision information system, where U is the universe; $A = C \cup D$ is the union of conditional attribute set C and decision attribute set D, and $C \cap D = \emptyset$; $V = \cup_{a \in A} V_a$ is the attribute value domain; $f : U \times A \to V$ is an information function, i.e., $\forall a \in A$, $x \in U$, that $f(x, a) \in V_a$, where $f(x, a)$ is the value of the object x under attribute a. R is the equivalence relation generated by C in the universe U. $[x]_R$ represents the equivalence class of x with regard to R, where $[x]_R = \{y \in U | f(y, a) = f(x, a), \forall a \in C\}$. U/C denotes the conditional partition consisting of equivalence classes generated by C in U and U/D denotes the decision partition consisting of decision classes generated by D in U. Symbol $|\cdot|$ denotes the cardinality of a set.

In Pawlak model, concepts are approximated by the upper and the lower approximations which consist of equivalence classes completely and possibly contained in concepts, respectively. Some scholars quantify the relationships between concepts and equivalence classes, then propose quantitative extension models. Yao [5] proposed decision-theoretic rough sets (DTRS) based on the relative quantitative information described by conditional probability and minimum Bayesian decision risk. For any concept X ($X \subseteq U$), in the DTRS model, the upper approximation is $\overline{R}_{(\alpha,\beta)}(X) = \cup\{[x]_R \mid P(X|[x]_R) > \beta, x \in U\}$ and the lower approximation is $\underline{R}_{(\alpha,\beta)}(X) = \cup\{[x]_R \mid P(X|[x]_R) \geq \alpha, x \in U\}$.

The basic idea of DTRS is to approximate concepts by equivalence classes with conditional probability in acceptable fault tolerance thresholds, where thresholds α and β are determined based on the minimum Bayesian decision risk and the loss function given by experts. Any object has two states with respect to a target concept. In each state, this object can be divided into positive, boundary and negative regions, which correspond to acceptance, non-commitment and rejection decisions, respectively. Let a_P, a_B and a_N denote the actions of classifying objects into positive region, boundary region and negative region, respectively. The six parameters of the loss function are briefly introduced. Let λ_{PP}, λ_{BP} and λ_{NP} denote the losses caused by taking actions a_P, a_B and a_N, respectively, when an object belongs to X; and λ_{PN}, λ_{BN} and λ_{NN} denote the losses incurred for taking the same actions when the object does not belong to X. This paper mainly studies the general case, namely $\beta < \alpha$, which can produce three rough regions and the corresponding three-way decisions. So the loss function concerned satisfies the two basic conditions $\lambda_{PP} \leq \lambda_{BP} < \lambda_{NP}$, $\lambda_{NN} \leq \lambda_{BN} < \lambda_{PN}$ and $(\lambda_{NP} - \lambda_{BP})(\lambda_{PN} - \lambda_{BN}) \geq (\lambda_{BP} - \lambda_{PP})(\lambda_{BN} - \lambda_{NN})$. The values of threshold parameters are $\alpha = \frac{\lambda_{PN}-\lambda_{BN}}{(\lambda_{PN}-\lambda_{BN})+(\lambda_{BP}-\lambda_{PP})}$ and $\beta = \frac{\lambda_{BN}-\lambda_{NN}}{(\lambda_{BN}-\lambda_{NN})+(\lambda_{NP}-\lambda_{BP})}$.

Considering that the relative quantitative information ignores the information differences of different equivalence classes containing the target concept to some extent, Li et al. [8] further proposed two types of double-quantitative decision-theoretic rough sets (Dq-DTRS) by cross-using relative and absolute quantitative information to characterize concepts in the upper and lower approximations.

In the first type of double-quantitative decision-theoretic rough set model (DqI-DTRS), the upper and lower approximations of any concept X ($X \subseteq U$) are

$$\overline{R}^{I}_{(\alpha,\beta)}(X) = \cup\{[x]_R \mid P(X|[x]_R) > \beta, x \in U\} \tag{2.1}$$

$$\underline{R}^{I}_{k}(X) = \cup\{[x]_R \mid \underline{g}([x]_R, X) \leq k, x \in U\} \tag{2.2}$$

where $0 \leq k \leq |U|$ and $0 \leq \beta = \frac{\lambda_{BN}-\lambda_{NN}}{(\lambda_{BN}-\lambda_{NN})+(\lambda_{NP}-\lambda_{BP})} \leq 1$. $P(X|[x]_R)$, called conditional probability [5,8], refers to the relative number of the elements belonging to X in equivalence class $[x]_R$, which can be calculated by $P(X|[x]_R) = \frac{|[x]_R \cap X|}{|[x]_R|}$. $\underline{g}([x]_R, X)$, called external grade [16], refers to the absolute number of the elements belonging to $[x]_R$ outside concept X, which can be calculated by $\underline{g}([x]_R, X) = |[x]_R| - |[x]_R \cap X|$. In the DqI-DTRS, the upper approximation of X is the union of equivalence classes whose conditional probability about X is greater than β and the lower approximation of X is the union of equivalence classes whose external grade about X is less than or equal to k. The positive, negative, upper boundary, lower boundary and boundary regions of X in the DqI-DTRS model are $pos^I(X) = \overline{R}^{I}_{(\alpha,\beta)}(X) \cap \underline{R}^{I}_{k}(X)$; $neg^I(X) = (\overline{R}^{I}_{(\alpha,\beta)}(X) \cup \underline{R}^{I}_{k}(X))^C$; $Ubn^I(X) = \overline{R}^{I}_{(\alpha,\beta)}(X) - \underline{R}^{I}_{k}(X)$; $Lbn^I(X) = \underline{R}^{I}_{k}(X) - \overline{R}^{I}_{(\alpha,\beta)}(X)$; $bn^I(X) = Ubn^I(X) \cup Lbn^I(X)$. Based on the positive, negative and boundary regions, we can obtain the acceptance, rejection and non-commitment decision rules, respectively.

In the second type of double-quantitative decision-theoretic rough set model (DqII-DTRS), the upper and lower approximations of X are

$$\overline{R}^{II}_{k}(X) = \cup\{[x]_R \mid \overline{g}([x]_R, X) > k, x \in U\} \tag{2.3}$$

$$\underline{R}^{II}_{(\alpha,\beta)}(X) = \cup\{[x]_R \mid P(X|[x]_R) \geq \alpha, x \in U\} \tag{2.4}$$

where $0 \leq k \leq |U|$ and $0 \leq \beta < \alpha = \frac{\lambda_{PN}-\lambda_{BN}}{(\lambda_{PN}-\lambda_{BN})+(\lambda_{BP}-\lambda_{PP})} \leq 1$. $\overline{g}([x]_R, X)$, called internal grade [16], refers to the absolute number of elements belonging to $[x]_R$ inside X, which can be calculated by $\overline{g}([x]_R, X) = |[x]_R \cap X|$. In the DqII-DTRS, the upper approximation of X is the union of equivalence classes whose internal grade about X is greater than k and the lower approximation of X is the union of equivalence classes whose conditional probability about X is greater than or equal to α. Similar to the DqI-DTRS model, the rough regions and the corresponding decision rules of X in the DqII-DTRS can be obtained based on $\overline{R}^{II}_{k}(X)$ and $\underline{R}^{II}_{(\alpha,\beta)}(X)$.

Table 1
The representations of related concepts with the variation of objects.

Time	t	$t+1$
System	$S = (U, A, V, f)$	$S' = (U', A, V', f')$
Universe	U	U'
Decision classes	$D_1, D_2, \ldots, D_n$	$D'_1, D'_2, \ldots, D'_{n'}$
Equivalence classes	$E_1, E_2, \ldots, E_m$	$E'_1, E'_2, \ldots, E'_{m'}$

3. Incremental approximation updating of Dq-DTRS with the sequential and batch insertion of objects

Concept approximations are the indispensable knowledge for feature selection and rule extraction of the Dq-DTRS models. This paper focuses on how to quickly obtain the approximations of DqI-DTRS and DqII-DTRS in dynamic decision information systems with the insertion and deletion of objects in sequences or in batches over time. We study the dynamic changes of objects from time t to time $t+1$. The basic information about the relevant concepts with the variation of objects at two moments is shown in Table 1.

In this section, we propose methods for updating the concept approximations of DqI-DTRS and DqII-DTRS when new objects are inserted sequentially or simultaneously over time.

Firstly, we study the updating methods when many objects are inserted sequentially at time $t+1$. Considering that the sequential approximation updating of many objects is the cumulative result of the updating one object after another, we mainly present the updating process of DqI-DTRS and DqII-DTRS when a new object x^+ is inserted. Then the approximation updating of many objects in the case of sequential insertion is realized by iteration loop.

The insertion of a new object will affect the granular structures of the previous conditional attribute set and decision attribute set, and then cause the variations of conditional probability, external grade and internal grade. Let $P = (p_{ij})_{n*m}$, $E = (e_{ij})_{n*m}$ and $I = (i_{ij})_{n*m}$ be the conditional probability matrix, external grade matrix and internal grade matrix of time t, respectively; $P' = (p'_{ij})_{n'*m'}$, $E' = (e'_{ij})_{n'*m'}$ and $I' = (i'_{ij})_{n'*m'}$ be the corresponding matrices of time $t+1$. It should be pointed out that $p_{ij} = P(D_i|E_j)$, $e_{ij} = \underline{g}(E_j, D_i)$, $i_{ij} = \overline{g}(E_j, D_i)$, $p'_{ij} = P(D'_i|E'_j)$, $e'_{ij} = \underline{g}(E'_j, D'_i)$ and $i'_{ij} = \overline{g}(E'_j, D'_i)$. Detailed variations of decision classes, equivalence classes, conditional probability, external grade and internal grade with the insertion of a new object are shown in the following lemma, where the other decision classes, equivalence classes, conditional probability, external grade and internal grade remain unchanged except that mentioned in the conclusions.

Lemma 3.1. *When x^+ is inserted at time $t+1$, we have the following conclusions:*

(1) *If $D'_i = D_i \cup \{x^+\}$ ($i \in \{1,2,\dots,n\}$) and $E'_j = E_j \cup \{x^+\}$ ($j \in \{1,2,\dots,m\}$), there are*
$p'_{1j} < p_{1j}, p'_{2j} < p_{2j}, \dots, p'_{i-1,j} < p_{i-1,j}, p'_{ij} > p_{ij}, p'_{i+1,j} < p_{i+1,j}, \dots, p'_{nj} < p_{nj}$;
$e'_{1j} > e_{1j}, e'_{2j} > e_{2j}, \dots, e'_{i-1,j} > e_{i-1,j}, e'_{i+1,j} > e_{i+1,j}, \dots, e'_{nj} > e_{nj}$;
$i'_{ij} > i_{ij}$.

(2) *If $D'_i = D_i \cup \{x^+\}$ ($i \in \{1,2,\dots,n\}$) and $E'_{m'} = E'_{m+1} = \{x^+\}$, there are*
$p'_{1,m+1} = p'_{2,m+1} = \cdots = p'_{i-1,m+1} = 0, p'_{i,m+1} = 1, p'_{i+1,m+1} = p'_{i+2,m+1} = \cdots = p'_{n,m+1} = 0$;
$e'_{1,m+1} = e'_{2,m+1} = \cdots = e'_{i-1,m+1} = 1, e'_{i,m+1} = 0, e'_{i+1,m+1} = e'_{i+2,m+1} = \cdots = e'_{n,m+1} = 1$;
$i'_{1,m+1} = i'_{2,m+1} = \cdots = i'_{i-1,m+1} = 0, i'_{i,m+1} = 1, i'_{i+1,m+1} = i'_{i+2,m+1} = \cdots = i'_{n,m+1} = 0$.

(3) *If $D'_{n'} = D'_{n+1} = \{x^+\}$ and $E'_j = E_j \cup \{x^+\}$ ($j \in \{1,2,\dots,m\}$), there are*
$p'_{n+1,1} = p'_{n+1,2} = \cdots = p'_{n+1,j-1} = 0, p'_{n+1,j} = \frac{1}{|E_j|+1}, p'_{n+1,j+1} = p'_{n+1,j+2} = \cdots = p'_{n+1,m} = 0$,
$p'_{1j} < p_{1j}, p'_{2j} < p_{2j}, \dots, p'_{nj} < p_{nj}$;
$e'_{1j} > e_{1j}, e'_{2j} > e_{2j}, \dots, e'_{nj} > e_{nj}, e'_{n+1,1} = |E_1|, e'_{n+1,2} = |E_2|, \dots, e'_{n+1,m} = |E_m|$;
$i'_{n+1,1} = 0, i'_{n+1,2} = 0, \dots, i'_{n+1,j-1} = 0, i'_{n+1,j} = 1, i'_{n+1,j+1} = 0, \dots, i'_{n+1,m} = 0$.

(4) *If $D'_{n'} = D'_{n+1} = \{x^+\}$ and $E'_{m'} = E'_{m+1} = \{x^+\}$, there are*
$p'_{n+1,1} = p'_{n+1,2} = \cdots = p'_{n+1,m} = 0, p'_{n+1,m+1} = 1, p'_{1,m+1} = p'_{2,m+1} = \cdots = p'_{n,m+1} = 0$;
$e'_{n+1,1} = |E_1|, e'_{n+1,2} = |E_2|, \dots, e'_{n+1,m} = |E_m|, e'_{n+1,m+1} = 0, e'_{1,m+1} = e'_{2,m+1} = \cdots = e'_{n,m+1} = 1$;
$i'_{n+1,1} = i'_{n+1,2} = \cdots = i'_{n+1,m} = 0, i'_{n+1,m+1} = 1, i'_{1,m+1} = i'_{2,m+1} = \cdots = i'_{n,m+1} = 0$.

Proof. By $D'_i = D_i \cup \{x^+\}$ and $E'_j = E_j \cup \{x^+\}$, there are $|E'_j| = |E_j| + 1$, $|D'_1 \cap E'_j| = |D_1 \cap E_j|, \cdots, |D'_{i-1} \cap E'_j| = |D_{i-1} \cap E_j|$, $|D'_i \cap E'_j| = |D_i \cap E_j| + 1$, $|D'_{i+1} \cap E'_j| = |D_{i+1} \cap E_j|, \cdots, |D'_n \cap E'_j| = |D_n \cap E_j|$. According to the definitions of the conditional probability, external grade and internal grade, the conclusions of case (1) are obvious. Other conclusions of cases (2), (3) and (4) can be obtained similarly. □

In the following, we give more intuitive results of the variations of conditional probability, external grade and internal grade at time $t+1$.

$$P'_{(1)} = \begin{pmatrix} p_{11} & \cdots & p_{1,j-1} & < p_{1j} & p_{1,j+1} & \cdots & p_{1m} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ p_{i-1,1} & \cdots & p_{i-1,j-1} & < p_{i-1,j} & p_{i-1,j+1} & \cdots & p_{i-1,m} \\ p_{i1} & \cdots & p_{i,j-1} & > p_{ij} & p_{i,j+1} & \cdots & p_{im} \\ p_{i+1,1} & \cdots & p_{i+1,j-1} & < p_{i+1j} & p_{i+1,j+1} & \cdots & p_{i+1,m} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ p_{n1} & \cdots & p_{n,j-1} & < p_{nj} & p_{n,j+1} & \cdots & p_{nm} \end{pmatrix}$$

$$P'_{(2)} = \left(\begin{array}{ccccccc|c} p_{11} & \cdots & p_{1,j-1} & p_{1j} & p_{1,j+1} & \cdots & p_{1m} & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ p_{i-1,1} & \cdots & p_{i-1,j-1} & p_{i-1,j} & p_{i-1,j+1} & \cdots & p_{i-1,m} & 0 \\ p_{i1} & \cdots & p_{i,j-1} & p_{ij} & p_{i,j+1} & \cdots & p_{im} & 1 \\ p_{i+1,1} & \cdots & p_{i+1,j-1} & p_{i+1,j} & p_{i+1,j+1} & \cdots & p_{i+1,m} & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ p_{n1} & \cdots & p_{n,j-1} & p_{nj} & p_{n,j+1} & \cdots & p_{nm} & 0 \end{array}\right)$$

$$E'_{(1)} = \begin{pmatrix} e_{11} & \cdots & e_{1,j-1} & > eg_{1j} & e_{1,j+1} & \cdots & e_{1m} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ e_{i-1,1} & \cdots & e_{i-1,j-1} & > e_{i-1,j} & e_{i-1,j+1} & \cdots & e_{i-1,m} \\ e_{i1} & \cdots & e_{i,j-1} & e_{ij} & e_{i,j+1} & \cdots & e_{im} \\ e_{i+1,1} & \cdots & e_{i+1,j-1} & > e_{i+1j} & e_{i+1,j+1} & \cdots & e_{i+1,m} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ e_{n1} & \cdots & e_{n,j-1} & > e_{nj} & e_{n,j+1} & \cdots & e_{nm} \end{pmatrix}$$

$$E'_{(2)} = \left(\begin{array}{ccccccc|c} e_{11} & \cdots & e_{1,j-1} & e_{1j} & e_{1,j+1} & \cdots & e_{1m} & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ e_{i-1,1} & \cdots & e_{i-1,j-1} & e_{i-1,j} & e_{i-1,j+1} & \cdots & e_{i-1,m} & 1 \\ e_{i1} & \cdots & e_{i,j-1} & e_{ij} & e_{i,j+1} & \cdots & e_{im} & 0 \\ e_{i+1,1} & \cdots & e_{i+1,j-1} & e_{i+1,j} & e_{i+1,j+1} & \cdots & e_{i+1,m} & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ e_{n1} & \cdots & e_{n,j-1} & e_{nj} & e_{n,j+1} & \cdots & e_{nm} & 1 \end{array}\right)$$

$$I'_{(1)} = \begin{pmatrix} i_{11} & \cdots & i_{1,j-1} & i_{1j} & i_{1,j+1} & \cdots & i_{1m} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ i_{i-1,1} & \cdots & i_{i-1,j-1} & i_{i-1,j} & i_{i-1,j+1} & \cdots & i_{i-1,m} \\ i_{i1} & \cdots & i_{i,j-1} & > i_{ij} & i_{i,j+1} & \cdots & i_{im} \\ i_{i+1,1} & \cdots & i_{i+1,j-1} & i_{i+1,j} & i_{i+1,j+1} & \cdots & i_{i+1,m} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ i_{n1} & \cdots & i_{n,j-1} & i_{nj} & i_{n,j+1} & \cdots & i_{nm} \end{pmatrix}$$

$$I'_{(2)} = \left(\begin{array}{ccccccc|c} i_{11} & \cdots & i_{1,j-1} & i_{1j} & i_{1,j+1} & \cdots & i_{1m} & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ i_{i-1,1} & \cdots & i_{i-1,j-1} & i_{i-1,j} & i_{i-1,j+1} & \cdots & i_{i-1,m} & 0 \\ i_{i1} & \cdots & i_{i,j-1} & i_{ij} & i_{i,j+1} & \cdots & i_{im} & 1 \\ i_{i+1,1} & \cdots & i_{i+1,j-1} & i_{i+1,j} & i_{i+1,j+1} & \cdots & i_{i+1,m} & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ i_{n1} & \cdots & i_{n,j-1} & i_{nj} & i_{n,j+1} & \cdots & i_{nm} & 0 \end{array}\right)$$

$$P'_{(3)} = \left(\begin{array}{ccccccc} p_{11} & \cdots & p_{1,j-1} & < p_{1j} & p_{1,j+1} & \cdots & p_{1m} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ p_{i-1,1} & \cdots & p_{i-1,j-1} & < p_{i-1,j} & p_{i-1,j+1} & \cdots & p_{i-1,m} \\ p_{i1} & \cdots & p_{i,j-1} & < p_{ij} & p_{i,j+1} & \cdots & p_{im} \\ p_{i+1,1} & \cdots & p_{i+1,j-1} & < p_{i+1,j} & p_{i+1,j+1} & \cdots & p_{i+1,m} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ p_{n1} & \cdots & p_{n,j-1} & < p_{nj} & p_{n,j+1} & \cdots & p_{nm} \\ \hline 0 & \cdots & 0 & \frac{1}{|E_j|+1} & 0 & \cdots & 0 \end{array}\right)$$

$$P'_{(4)} = \left(\begin{array}{ccccccc|c} p_{11} & \cdots & p_{1,j-1} & p_{1j} & p_{1,j+1} & \cdots & p_{1m} & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ p_{i-1,1} & \cdots & p_{i-1,j-1} & p_{i-1,j} & p_{i-1,j+1} & \cdots & p_{i-1,m} & 0 \\ p_{i1} & \cdots & p_{i,j-1} & p_{ij} & p_{i,j+1} & \cdots & p_{im} & 0 \\ p_{i+1,1} & \cdots & p_{i+1,j-1} & p_{i+1,j} & p_{i+1,j+1} & \cdots & p_{i+1,m} & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ p_{n1} & \cdots & p_{n,j-1} & p_{nj} & p_{n,j+1} & \cdots & p_{nm} & 0 \\ \hline 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 \end{array}\right)$$

$$E'_{(3)} = \left(\begin{array}{ccccccc} e_{11} & \cdots & e_{1,j-1} & > eg_{1j} & e_{1,j+1} & \cdots & e_{1m} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ e_{i-1,1} & \cdots & e_{i-1,j-1} & > eg_{i-1,j} & e_{i-1,j+1} & \cdots & e_{i-1,m} \\ e_{i1} & \cdots & e_{i,j-1} & > eg_{ij} & e_{i,j+1} & \cdots & e_{im} \\ e_{i+1,1} & \cdots & e_{i+1,j-1} & > eg_{i+1,j} & e_{i+1,j+1} & \cdots & e_{i+1,m} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ e_{n1} & \cdots & e_{n,j-1} & > eg_{nj} & e_{n,j+1} & \cdots & e_{nm} \\ \hline |E_1| & \cdots & |E_{j-1}| & |E_j| & |E_{j+1}| & \cdots & |E_m| \end{array}\right)$$

$$E'_{(4)} = \left(\begin{array}{ccccccc|c} e_{11} & \cdots & e_{1,j-1} & e_{1j} & e_{1,j+1} & \cdots & e_{1m} & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ e_{i-1,1} & \cdots & e_{i-1,j-1} & e_{i-1,j} & e_{i-1,j+1} & \cdots & e_{i-1,m} & 1 \\ e_{i1} & \cdots & e_{i,j-1} & e_{ij} & e_{i,j+1} & \cdots & e_{im} & 1 \\ e_{i+1,1} & \cdots & e_{i+1,j-1} & e_{i+1,j} & e_{i+1,j+1} & \cdots & e_{i+1,m} & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ e_{n1} & \cdots & e_{n,j-1} & e_{nj} & e_{n,j+1} & \cdots & e_{nm} & 1 \\ \hline |E_1| & \cdots & |E_{j-1}| & |E_j| & |E_{j+1}| & \cdots & |E_m| & 0 \end{array}\right)$$

$$I'_{(3)} = \left(\begin{array}{ccccccc} i_{11} & \cdots & i_{1,j-1} & i_{1j} & i_{1,j+1} & \cdots & i_{1m} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ i_{i-1,1} & \cdots & i_{i-1,j-1} & i_{i-1,j} & i_{i-1,j+1} & \cdots & i_{i-1,m} \\ i_{i1} & \cdots & i_{i,j-1} & i_{ij} & i_{i,j+1} & \cdots & i_{im} \\ i_{i+1,1} & \cdots & i_{i+1,j-1} & i_{i+1,j} & i_{i+1,j+1} & \cdots & i_{i+1,m} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ i_{n1} & \cdots & i_{n,j-1} & i_{nj} & i_{n,j+1} & \cdots & i_{nm} \\ \hline 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{array}\right)$$

$$I'_{(4)}=\left(\begin{array}{ccccccc:c} i_{11} & \cdots & i_{1,j-1} & i_{1j} & i_{1,j+1} & \cdots & i_{1m} & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ i_{i-1,1} & \cdots & i_{i-1,j-1} & i_{i-1,j} & i_{i-1,j+1} & \cdots & i_{i-1,m} & 0\\ i_{i1} & \cdots & i_{i,j-1} & i_{ij} & i_{i,j+1} & \cdots & i_{im} & 0\\ i_{i+1,1} & \cdots & i_{i+1,j-1} & i_{i+1,j} & i_{i+1,j+1} & \cdots & i_{i+1,m} & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ i_{n1} & \cdots & i_{n,j-1} & i_{nj} & i_{n,j+1} & \cdots & i_{nm} & 0\\ \hdashline 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 \end{array}\right)$$

From the above $P'_{(1)-(4)}, E'_{(1)-(4)}, I'_{(1)-(4)}$, we can see the information association and the change between time t and $t+1$. In the $P'_{(1)-(4)}, E'_{(1)-(4)}, I'_{(1)-(4)}$, the information on the right or the bottom of the dotted line of the matrix is new probability or grade information. In cases (1)–(4), when updating the upper and lower approximations of original decision classes, we are mainly concerned with the information of the jth or $(m+1)$th column of $P'_{(1)-(4)}, E'_{(1)-(4)}, I'_{(1)-(4)}$. In cases (3)–(4), when calculating the approximations of a new decision class, we can only recalculate them according to formulas (2.1)–(2.4) because there is no prior knowledge.

3.1. The updating mechanisms for the concept approximations of DqI-DTRS with the insertion of a new object

We first propose the updating mechanisms for the approximations of DqI-DTRS when a new object x^+ is inserted at time $t+1$. For any decision class D_i ($D_i \subseteq U$), we present the update process of the approximations of D'_i.

Case 1: The new object belongs to one original decision class, namely $\exists x_i \in U, \forall d \in D$, s.t. $f(x_i, d) = f(x^+, d)$.

Proposition 3.1.1. *When $|[x^+]_R| = 1$, for the upper and lower approximations of D'_i, we have the following properties:*

(1) $x^+ \notin D'_i$, $\overline{R}^{I}_{(\alpha,\beta)}(D'_i) = \overline{R}^{I}_{(\alpha,\beta)}(D_i)$.
(2) $x^+ \notin D'_i$, if $k > 0$, then $\underline{R}^{I}_{k}(D'_i) = \underline{R}^{I}_{k}(D_i) \cup \{x^+\}$; otherwise, $\underline{R}^{I}_{k}(D'_i) = \underline{R}^{I}_{k}(D_i)$.
(3) $x^+ \in D'_i$, $\overline{R}^{I}_{(\alpha,\beta)}(D'_i) = \overline{R}^{I}_{(\alpha,\beta)}(D_i) \cup \{x^+\}$, $\underline{R}^{I}_{k}(D'_i) = \underline{R}^{I}_{k}(D_i) \cup \{x^+\}$.

Proof. (1) From $|[x^+]_R| = 1$ and $x^+ \notin D'_i$, there are $[x^+]_R = \{x^+\}$ and $D'_i = D_i$. By $P(D'_i|[x^+]_R) = 0$, we have $x^+ \notin \overline{R}^{I}_{(\alpha,\beta)}(D'_i)$. So the conclusion $\overline{R}^{I}_{(\alpha,\beta)}(D'_i) = \overline{R}^{I}_{(\alpha,\beta)}(D_i)$ is true.

(2) According to $[x^+]_R = \{x^+\}$ and $D'_i = D_i$, there is $\underline{g}([x^+]_R, D'_i) = 1$. If $k > 0$, then $x^+ \in \underline{g}([x^+]_R, D'_i)$. So $\underline{R}^{I}_{k}(D'_i) = \underline{R}^{I}_{k}(D_i) \cup \{x^+\}$. If $k = 0$, $x^+ \notin \underline{g}([x^+]_R, D'_i)$. Therefore, $\underline{R}^{I}_{k}(D'_i) = \underline{R}^{I}_{k}(D_i)$.

(3) From $|[x^+]_R| = 1$ and $x^+ \in D'_i$, there are $[x^+]_R = \{x^+\}$ and $D'_i = D_i \cup \{x^+\}$. By $P(D'_i|[x^+]_R) = 1$ and $\underline{g}([x^+]_R, D'_i) = 0$, we can get $x^+ \in \overline{R}^{I}_{(\alpha,\beta)}(D'_i)$ and $x^+ \in \underline{R}^{I}_{k}(D'_i)$. Therefore, $\overline{R}^{I}_{(\alpha,\beta)}(D'_i) = \overline{R}^{I}_{(\alpha,\beta)}(D_i) \cup \{x^+\}$ and $\underline{R}^{I}_{k}(D'_i) = \underline{R}^{I}_{k}(D_i) \cup \{x^+\}$.

Thus, Proposition 3.1.1 is proved. □

According to Proposition 3.1.1, when the equivalence class of the new object is a singleton set, there are (1) The upper and lower approximations of the decision class to which the new object belongs can be updated directly; (2) The lower approximations of those decision classes to which the new object does not belong also need to be updated as long as the grade parameter k is greater than 0.

When the equivalence class $[x^+]_R$ is not a singleton set, we suppose that x^+ belongs to one original equivalence class E_j. At time $t+1$, this equivalence class E_j becomes E'_j, where $E'_j = E_j \cup \{x^+\}$.

Proposition 3.1.2. *When $|[x^+]_R| > 1$, $x^+ \in E_j$, if $x^+ \in D'_i$, the following properties for $\overline{R}^{I}_{(\alpha,\beta)}(D'_i)$ and $\underline{R}^{I}_{k}(D'_i)$ hold.*

(1) When $E_j \subseteq \overline{R}^{I}_{(\alpha,\beta)}(D_i)$, there is $\overline{R}^{I}_{(\alpha,\beta)}(D'_i) = \overline{R}^{I}_{(\alpha,\beta)}(D_i) \cup \{x^+\}$.
(2) When $E_j \not\subset \overline{R}^{I}_{(\alpha,\beta)}(D_i)$, if $P(D'_i|E'_j) > \beta$, then $\overline{R}^{I}_{(\alpha,\beta)}(D'_i) = \overline{R}^{I}_{(\alpha,\beta)}(D_i) \cup E'_j$; otherwise, $\overline{R}^{I}_{(\alpha,\beta)}(D'_i) = \overline{R}^{I}_{(\alpha,\beta)}(D_i)$.
(3) When $E_j \subseteq \underline{R}^{I}_{k}(D_i)$, there is $\underline{R}^{I}_{k}(D'_i) = \underline{R}^{I}_{k}(D_i) \cup \{x^+\}$.
(4) When $E_j \not\subset \underline{R}^{I}_{k}(D_i)$, there is $\underline{R}^{I}_{k}(D'_i) = \underline{R}^{I}_{k}(D_i)$.

Proof. By $|[x^+]_R| > 1$, $x^+ \in E_j$ and $x^+ \in D'_i$, there are $E'_j = E_j \cup \{x^+\}$ and $D'_i = D_i \cup \{x^+\}$.

(1) When $E_j \subseteq \overline{R}^{I}_{(\alpha,\beta)}(D_i)$, there is $P(D_i|E_j) > \beta$. By $P(D'_i|E'_j) = \frac{|E'_j \cap D'_i|}{|E'_j|} = \frac{|E_j \cap D_i|+1}{|E_j|+1} > P(D_i|E_j)$, there is $E'_j \subseteq \overline{R}^{I}_{(\alpha,\beta)}(D'_i)$. It is true that $\overline{R}^{I}_{(\alpha,\beta)}(D'_i) = \overline{R}^{I}_{(\alpha,\beta)}(D_i) \cup \{x^+\}$.

(2) When $E_j \not\subset \overline{R}^{I}_{(\alpha,\beta)}(D_i)$, there is $P(D_i|E_j) \le \beta$. By $P(D'_i|E'_j) > P(D_i|E_j)$, if $P(E'_j, D'_i) > \beta$, we obtain $E'_j \subseteq \overline{R}^{I}_{(\alpha,\beta)}(D'_i)$. So $\overline{R}^{I}_{(\alpha,\beta)}(D'_i) = \overline{R}^{I}_{(\alpha,\beta)}(D_i) \cup E'_j$. If $P(E'_j, D'_i) \le \beta$, then $E'_j \not\subset \overline{R}^{I}_{(\alpha,\beta)}(D_i)$. Therefore, $\overline{R}^{I}_{(\alpha,\beta)}(D'_i) = \overline{R}^{I}_{(\alpha,\beta)}(D_i)$.

(3) When $E_j \subseteq \underline{R}^{I}_{k}(D_i)$, we can get $\underline{g}(E_j, D_i) \le k$. By $\underline{g}(E'_j, D'_i) = |E'_j| - |E'_j \cap D'_i| = |E_j| - |E_j \cap D_i| = \underline{g}(E_j, D_i)$, there is $E'_j \subseteq \underline{R}^{I}_{k}(D'_i)$. Therefore, $\underline{R}^{I}_{k}(D'_i) = \underline{R}^{I}_{k}(D_i) \cup \{x^+\}$

(4) By $E_j \not\subset \underline{R}^{I}_{k}(D_i)$ and $\underline{g}(E_j, D_i) = \underline{g}(E'_j, D'_i)$, it is true that $\underline{R}^{I}_{k}(D'_i) = \underline{R}^{I}_{k}(D_i)$.

Thus, the proof of Proposition 3.1.2 is finished. □

According to Proposition 3.1.2, when the equivalence class of the new object is not a singleton set and this new object belongs to one original decision class, there are (1) When the equivalence class to which this new object belongs is contained in the upper (lower) approximation of the original decision class, the upper (lower) approximation of the corresponding current decision class can be updated directly; (2) When the equivalence class is not contained in the upper approximation of the original decision class, the upper approximation of the corresponding current decision class needs to be updated as long as the current conditional probability is greater than β.

Proposition 3.1.3. *When $|[x^+]_R| > 1$, $x^+ \in E_j$, if $x^+ \notin D'_i$, the following properties for $\overline{R}^{I}_{(\alpha,\beta)}(D'_i)$ and $\underline{R}^{I}_{k}(D'_i)$ hold.*

(1) When $E_j \subseteq \overline{R}^{I}_{(\alpha,\beta)}(D_i)$, if $P(D'_i|E'_j) > \beta$, then $\overline{R}^{I}_{(\alpha,\beta)}(D'_i) = \overline{R}^{I}_{(\alpha,\beta)}(D_i) \cup \{x^+\}$; otherwise, $\overline{R}^{I}_{(\alpha,\beta)}(D'_i) = \overline{R}^{I}_{(\alpha,\beta)}(D_i) - E_j$.
(2) When $E_j \not\subset \overline{R}^{I}_{(\alpha,\beta)}(D_i)$, there is $\overline{R}^{I}_{(\alpha,\beta)}(D'_i) = \overline{R}^{I}_{(\alpha,\beta)}(D_i)$.
(3) When $E_j \subseteq \underline{R}^{I}_{k}(D_i)$, if $\underline{g}(E'_j, D'_i) \le k$, then $\underline{R}^{I}_{k}(D'_i) = \underline{R}^{I}_{k}(D_i) \cup \{x^+\}$; otherwise, $\underline{R}^{I}_{k}(D'_i) = \underline{R}^{I}_{k}(D_i) - E_j$.
(4) When $E_j \not\subset \underline{R}^{I}_{k}(D_i)$, there is $\underline{R}^{I}_{k}(D'_i) = \underline{R}^{I}_{k}(D_i)$.

Proof. By $|[x^+]_R| > 1$, $x^+ \in E_j$ and $x^+ \notin D'_i$, there are $E'_j = E_j \cup \{x^+\}$ and $D'_i = D_i$. We further get two conclusions: $P(D'_i|E'_j) = \frac{|E_j \cap D_i|}{|E_j|+1} < P(D_i|E_j)$ and $\underline{g}(E'_j, D'_i) = |E_j| - |E_j \cap D_i| + 1 > \underline{g}(E_j, D_i)$.

(1) When $E_j \subseteq \overline{R}^{I}_{(\alpha,\beta)}(D_i)$, if $P(E'_j, D'_i) > \beta$, then $E'_j \subseteq \overline{R}^{I}_{(\alpha,\beta)}(D'_i)$. So $\overline{R}^{I}_{(\alpha,\beta)}(D'_i) = \overline{R}^{I}_{(\alpha,\beta)}(D_i) \cup E'_j = \overline{R}^{I}_{(\alpha,\beta)}(D_i) \cup \{x^+\}$. On the contrary, $E'_j \not\subset \overline{R}^{I}_{(\alpha,\beta)}(D'_i)$ is true. By $E_i \subseteq \overline{R}^{I}_{(\alpha,\beta)}(D_i)$ and $E'_j \not\subset \overline{R}^{I}_{(\alpha,\beta)}(D'_i)$, we can get $\overline{R}^{I}_{(\alpha,\beta)}(D'_i) = \overline{R}^{I}_{(\alpha,\beta)}(D_i) - E_j$.

(2) By $E_j \not\subset \overline{R}^{I}_{(\alpha,\beta)}(D_i)$ and $P(D'_i|E'_j) < P(D_i|E_j)$, we obtain $E'_j \not\subset \overline{R}^{I}_{(\alpha,\beta)}(D'_i)$. Therefore, $\overline{R}^{I}_{(\alpha,\beta)}(D'_i) = \overline{R}^{I}_{(\alpha,\beta)}(D_i)$.

(3) By $E_j \subseteq \underline{R}^{I}_{k}(D_i)$ and $\underline{g}(E'_j, D'_i) \le k$, we can get $E'_j \subseteq \underline{R}^{I}_{k}(D'_i)$. So $\underline{R}^{I}_{k}(D'_i) = \underline{R}^{I}_{k}(D_i) \cup E'_j = \underline{R}^{I}_{k}(D_i) \cup \{x^+\}$. On the contrary, $E'_j \not\subset \underline{R}^{I}_{k}(D'_i)$. By $E_j \subseteq \underline{R}^{I}_{k}(D_i)$, it is true that $\underline{R}^{I}_{k}(D'_i) = \underline{R}^{I}_{k}(D_i) - E'_j = \underline{R}^{I}_{k}(D_i) - E_j$.

Table 2
A decision information system of time t.

U	a_1	a_2	a_3	a_4	d
x_1	1	0	1	1	0
x_2	1	1	1	1	0
x_3	1	0	1	1	1
x_4	0	1	0	1	1
x_5	1	1	1	0	1
x_6	1	0	1	1	0
x_7	1	1	1	1	0
x_8	1	1	1	0	1

Table 3
The system with the insertion of an object.

U	a_1	a_2	a_3	a_4	d
x_1	1	0	1	1	0
x_2	1	1	1	1	0
x_3	1	0	1	1	1
x_4	0	1	0	1	1
x_5	1	1	1	0	1
x_6	1	0	1	1	0
x_7	1	1	1	1	0
x_8	1	1	1	0	1
$\mathbf{x_9}$	**1**	**1**	**1**	**0**	**0**

(4) According to $E_j \not\subset \underline{R}^I_k(D_i)$ and $\underline{g}(E'_j, D'_i) > \underline{g}(E_j, D_i)$, there is $E'_j \not\subset \underline{R}^I_k(D'_i)$. Therefore, $\underline{R}^I_k(D'_i) = \underline{R}^I_k(D_i)$. □

From Proposition 3.1.3, when the equivalence class of the new object is not a singleton set and this new object does not belong to one original decision class, the upper (lower) approximation of the corresponding current decision class needs to be updated only when the equivalence class is contained in the upper (lower) approximation of the original decision class.

By deeply analyzing the relationships among Propositions 3.1.1–3.1.3, we find that Proposition 3.1.1 is a special case of Propositions 3.1.2–3.1.3. In the following discussions, no more attention is paid to whether the equivalence class of a new object is a singleton set.

Case 2: A new decision class appears at time $t+1$, namely $D^+ = \{x^+\}$.

Proposition 3.1.4. *For x^+, if $\forall x_i \in U, \exists d \in D$, s.t. $f(x_i, d) \neq f(x^+, d)$, there are*

(1) If $P(D^+|[x^+]_R) > \beta$, then $\overline{R}^I_{(\alpha,\beta)}(D^+) = [x^+]_R$; otherwise, $\overline{R}^I_{(\alpha,\beta)}(D^+) = \emptyset$.
(2) For $j \in \{1, 2, \ldots, m'\}$, if $\underline{g}(E'_j, D^+) \leq k$, then $\underline{R}^I_k(D^+) = \cup\{E'_j\}$.

Proof. According to the definition of lower and upper approximations of DqI-DTRS (formulas (2.1) and (2.2)), the above conclusions are obvious.

Example 1. A decision information system $S = (U, A = C \cup D, V, f)$ at time t is shown in Table 2, where the universe $U = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$, the conditional attribute set $C = \{a_1, a_2, a_3, a_4\}$ and the decision attribute set $D = \{d\}$. At time $t+1$, a new object x_9 is inserted and its detailed information is shown in Table 3. We elaborate on the incremental approximation updating mechanisms of DqI-DTRS with the insertion of an object, where $\beta = \frac{1}{3}$ and $k = 1$.

At time t, we know that $U/D = \{D_1 = \{x_1, x_2, x_6, x_7\}, D_2 = \{x_3, x_4, x_5, x_8\}\}$ and $U/C = \{E_1 = \{x_1, x_3, x_6\}, E_2 = \{x_2, x_7\}, E_3 = \{x_4\}, E_4 = \{x_5, x_8\}\}$ from Table 2. Moreover, there are

$$P(D_1|E_1) = \frac{2}{3}, P(D_1|E_2) = 1, P(D_1|E_3) = 0, P(D_1|E_4) = 0,$$
$$P(D_2|E_1) = \frac{1}{3}, P(D_2|E_2) = 0, P(D_2|E_3) = 1, P(D_2|E_4) = 1;$$
$$\underline{g}(E_1, D_1) = 1, \underline{g}(E_2, D_1) = 0, \underline{g}(E_3, D_1) = 1, \underline{g}(E_4, D_1) = 2,$$
$$\underline{g}(E_1, D_2) = 2, \underline{g}(E_2, D_2) = 2, \underline{g}(E_3, D_2) = 0, \underline{g}(E_4, D_2) = 0.$$

According to formulas (2.1) and (2.2), there are $\overline{R}^I_{(\alpha,\beta)}(D_1) = E_1 \cup E_2$, $\underline{R}^I_k(D_1) = E_1 \cup E_2 \cup E_3$, $\overline{R}^I_{(\alpha,\beta)}(D_2) = E_3 \cup E_4$, $\underline{R}^I_k(D_2) = E_3 \cup E_4$.

At time $t+1$, we know that $D'_1 = D_1 \cup \{x_9\} = \{x_1, x_2, x_6, x_7, x_9\}$, $D'_2 = D_2 = \{x_3, x_4, x_5, x_8\}$ and $E'_1 = E_1, E'_2 = E_2, E'_3 = E_3, E'_4 = E_4 \cup \{x_9\} = \{x_5, x_8, x_9\}$ from Table 3.

By $x_9 \in E_4$ and $x_9 \in D'_1$, we calculate the approximations of D'_1 according to Proposition 3.1.2. Because $E_4 \not\subset \overline{R}^I_{(\alpha,\beta)}(D_1)$ and $P(D'_1|E'_4) = \frac{1}{3} \ngtr \beta$, there is $\overline{R}^I_{(\alpha,\beta)}(D'_1) = \overline{R}^I_{(\alpha,\beta)}(D_1)$ by the conclusion (2) of Proposition 3.1.2. Because $E_4 \not\subset \underline{R}^I_k(D_1)$, there is $\underline{R}^I_k(D'_1) = \underline{R}^I_k(D_1)$ by the conclusion (4) of Proposition 3.1.2.

By $x_9 \in E_4$ and $x_9 \notin D'_2$, we get the approximations of D'_2 according to Proposition 3.1.3. Because $E_4 \subset \overline{R}^I_{(\alpha,\beta)}(D_2)$ and $P(D'_2|E'_4) = \frac{2}{3} > \beta$, there is $\overline{R}^I_{(\alpha,\beta)}(D'_2) = \overline{R}^I_{(\alpha,\beta)}(D_2) \cup \{x_9\}$ by the conclusion (1) of Proposition 3.1.3. Because $E_4 \subset \underline{R}^I_k(D_2)$ and $\underline{g}(E'_4, D'_2) = 1 \leq k$, there is $\underline{R}^I_k(D'_2) = \underline{R}^I_k(D_2) \cup \{x_9\}$ by the conclusion (3) of Proposition 3.1.3.

3.2. The updating mechanisms for the concept approximations of DqII-DTRS with the insertion of a new object

In the following, we study the updating mechanisms of the upper and lower approximations of DqII-DTRS when a new object x^+ is inserted at time $t+1$.

Case 1: The new object belongs to one original decision class, namely $\exists x_i \in U, \forall d \in D$, s.t. $f(x_i, d) = f(x^+, d)$.

Based on the previous research experience of Section 3.1, we directly assume that x^+ belongs to one equivalence class E_j of the decision information S. At time $t+1$, the equivalence class E_j becomes E'_j, where $E'_j = E_j \cup \{x^+\}$.

Proposition 3.2.1. *Let $x^+ \in E_j$, if $x^+ \in D'_i$, the following properties for $\overline{R}^{II}_k(D'_i)$ and $\underline{R}^{II}_{(\alpha,\beta)}(D'_i)$ hold.*

(1) When $E_j \subseteq \overline{R}^{II}_k(D_i)$, there is $\overline{R}^{II}_k(D'_i) = \overline{R}^{II}_k(D_i) \cup \{x^+\}$.
(2) When $E_j \not\subset \overline{R}^{II}_k(D_i)$, if $\overline{g}(E'_j, D'_i) > k$, then $\overline{R}^{II}_k(D'_i) = \overline{R}^{II}_k(D_i) \cup E'_j$; otherwise, $\overline{R}^{II}_k(D'_i) = \overline{R}^{II}_k(D_i)$.
(3) When $E_j \subseteq \underline{R}^{II}_{(\alpha,\beta)}(D_i)$, there is $\underline{R}^{II}_{(\alpha,\beta)}(D'_i) = \underline{R}^{II}_{(\alpha,\beta)}(D_i) \cup \{x^+\}$.
(4) When $E_j \not\subset \underline{R}^{II}_{(\alpha,\beta)}(D_i)$, if $P(D'_i|E'_j) \geq \alpha$, then $\underline{R}^{II}_{(\alpha,\beta)}(D'_i) = \underline{R}^{II}_{(\alpha,\beta)}(D_i) \cup E'_j$; otherwise, $\underline{R}^{II}_{(\alpha,\beta)}(D'_i) = \underline{R}^{II}_{(\alpha,\beta)}(D_i)$.

Proof. By $E'_j = E_j \cup \{x^+\}$ and $D'_i = D_i \cup \{x^+\}$, we can get $\overline{g}(E'_j, D'_i) > \overline{g}(E_j, D_i)$ and $P(D'_i|E'_j) > P(D_i|E_j)$.

(1) When $E_j \subseteq \overline{R}^{II}_k(D_i)$, there is $\overline{g}(E_j, D_i) > k$. By $\overline{g}(E'_j, D'_i) > \overline{g}(E_j, D_i) > k$, we can get $E'_j \subseteq \overline{R}^{II}_k(D'_i)$. Therefore, $\overline{R}^{II}_k(D'_i) = \overline{R}^{II}_k(D_i) \cup \{x^+\}$.

(2) According to $E_j \not\subset \overline{R}^{II}_k(D_i)$ and $\overline{g}(E'_j, D'_i) > k$, we can get $E'_j \subseteq \overline{R}^{II}_k(D'_i)$ and $\overline{R}^{II}_k(D'_i) = \overline{R}^{II}_k(D_i) \cup E'_j$. In another case, by $E_j \not\subset \overline{R}^{II}_k(D_i)$ and $E'_j \not\subset \overline{R}^{II}_k(D'_i)$, we can obtain $\overline{R}^{II}_k(D'_i) = \overline{R}^{II}_k(D_i)$.

(3) When $E_j \subseteq \underline{R}^{II}_{(\alpha,\beta)}(D_i)$, there is $P(D_i|E_j) \geq \alpha$. By $P(D'_i|E'_j) > P(D_i|E_j)$, so we can get $E'_j \subseteq \underline{R}^{II}_{(\alpha,\beta)}(D'_i)$ and $\underline{R}^{II}_{(\alpha,\beta)}(D'_i) = \underline{R}^{II}_{(\alpha,\beta)}(D_i) \cup \{x^+\}$.

(4) When $E_j \not\subseteq \underline{R}^{II}_{(\alpha,\beta)}(D_i)$, if $P(D'_i|E'_j) \geq \alpha$, there is $E'_j \subseteq \underline{R}^{II}_{(\alpha,\beta)}(D'_i)$. Therefore, $\underline{R}^{II}_{(\alpha,\beta)}(D'_i) = \underline{R}^{II}_{(\alpha,\beta)}(D_i) \cup E'_j$. On the contrary, by $E_j \not\subset \overline{R}^{II}_k(D_i)$ and $E'_j \not\subset \overline{R}^{II}_k(D'_i)$, we get $\underline{R}^{II}_{(\alpha,\beta)}(D'_i) = \underline{R}^{II}_{(\alpha,\beta)}(D_i)$. □

From Proposition 3.2.1, we first consider a prerequisite, namely the new object belongs to one original decision class. When the original equivalence class to which this new object belongs is contained in the upper (lower) approximation of the original decision class, the upper (lower) approximation of the corresponding current decision class can be updated directly. When this original equivalence class is not contained in the upper (lower) approximation of the original decision class, we need to further determine whether the current internal grade (conditional probability) meets the threshold requirement. Then we decide whether to update the upper (lower) approximation.

Proposition 3.2.2. *Let $x^+ \in E_j$, if $x^+ \notin D'_i$, the following properties for $\overline{R}^{II}_k(D'_i)$ and $\underline{R}^{II}_{(\alpha,\beta)}(D'_i)$ hold.*

(1) When $E_j \subseteq \overline{R}^{II}_k(D_i)$, there is $\overline{R}^{II}_k(D'_i) = \overline{R}^{II}_k(D_i) \cup \{x^+\}$.
(2) When $E_j \not\subset \overline{R}^{II}_k(D_i)$, there is $\overline{R}^{II}_k(D'_i) = \overline{R}^{II}_k(D_i)$.
(3) When $E_j \subseteq \underline{R}^{II}_{(\alpha,\beta)}(D_i)$, if $P(D'_i|E'_j) \geq \alpha$, then $\underline{R}^{II}_{(\alpha,\beta)}(D'_i) = \underline{R}^{II}_{(\alpha,\beta)}(D_i) \cup \{x^+\}$; otherwise, $\underline{R}^{II}_{(\alpha,\beta)}(D'_i) = \underline{R}^{II}_{(\alpha,\beta)}(D_i) - E_j$.
(4) When $E_j \not\subset \underline{R}^{II}_{(\alpha,\beta)}(D_i)$, there is $\underline{R}^{II}_{(\alpha,\beta)}(D'_i) = \underline{R}^{II}_{(\alpha,\beta)}(D_i)$.

Proof. By $E'_j = E_j \cup \{x^+\}$ and $D'_i = D_i$, we can get $\overline{g}(E'_j, D'_i) = \overline{g}(E_j, D_i)$ and $P(D'_i|E'_j) < P(D_i|E_j)$. The conclusions about $\overline{R}^{II}_k(D'_i)$ and $\underline{R}^{II}_{(\alpha,\beta)}(D'_i)$ can be obtained by formulas (2.3) and (2.4) and two expressions $\overline{g}(E'_j, D'_i) = \overline{g}(E_j, D_i)$ and $P(D'_i|E'_j) < P(D_i|E_j)$. This proof is similar to the proof of Proposition 3.2.1. □

From Proposition 3.2.2, the prerequisite is that the new object does not belong to one original decision class. When the original equivalence class to which this new object belongs is contained in the upper (lower) approximation of the original decision class, the upper (lower) approximation of the corresponding current decision class needs to be updated.

Case 2: A new decision class appears at time $t+1$, namely $D^+ = \{x^+\}$.

Proposition 3.2.3. *For x^+, if $\forall x_i \in U$, $\exists d \in D$, s.t. $f(x_i, d) \neq f(x^+, d)$, then the following properties for $\overline{R}^{II}_k(D^+)$ and $\underline{R}^{II}_{(\alpha,\beta)}(D^+)$ hold, where $D^+ = \{x^+\}$.*

(1) If $\overline{g}([x^+]_R, D^+) > k$, then $\overline{R}^{II}_k(D^+) = [x^+]_R$; otherwise, $\overline{R}^{II}_k(D^+) = \emptyset$.
(2) If $P(D^+|[x^+]_R) \geq \alpha$, then $\underline{R}^{II}_{(\alpha,\beta)}(D^+) = [x^+]_R$; otherwise, $\underline{R}^{II}_{(\alpha,\beta)}(D^+) = \emptyset$.

Proof. According to the definition of lower and upper approximations of DqII-DTRS (formulas (2.3) and (2.4)), we know that only equivalence classes that intersect with D^+ are not empty may be included in the approximations $\overline{R}^{II}_k(D^+)$ and $\underline{R}^{II}_{(\alpha,\beta)}(D^+)$. So the above conclusions are obvious. □

Example 2 (*Continuation of Example 1*). We elaborate on the incremental approximation updating mechanisms of DqII-DTRS with the insertion of an object, where the parameters of DqII-DTRS are $k = 1$ and $\alpha = \frac{2}{3}$.

At time t, we get that the internal grade and conditional probability are

$$\overline{g}(E_1, D_1) = 2, \overline{g}(E_2, D_1) = 2, \overline{g}(E_3, D_1) = 0, \overline{g}(E_4, D_1) = 0,$$
$$\overline{g}(E_1, D_2) = 1, \overline{g}(E_2, D_2) = 0, \overline{g}(E_3, D_2) = 1, \overline{g}(E_4, D_2) = 2;$$
$$P(D_1|E_1) = \frac{2}{3}, P(D_1|E_2) = 1, P(D_1|E_3) = 0, P(D_1|E_4) = 0,$$
$$P(D_2|E_1) = \frac{1}{3}, P(D_2|E_2) = 0, P(D_2|E_3) = 1, P(D_2|E_4) = 1.$$

By formulas (2.3) and (2.4), there are $\overline{R}^{II}_k(D_1) = E_1 \cup E_2$, $\underline{R}^{II}_{(\alpha,\beta)}(D_1) = E_1 \cup E_2$, $\overline{R}^{II}_k(D_2) = E_4$, $\underline{R}^{II}_{(\alpha,\beta)}(D_2) = E_3 \cup E_4$.

At time $t+1$, we know that $D'_1 = D_1 \cup \{x_9\} = \{x_1, x_2, x_6, x_7, x_9\}$, $D'_2 = D_2 = \{x_3, x_4, x_5, x_8\}$ and $E'_1 = E_1, E'_2 = E_2, E'_3 = E_3, E'_4 = E_4 \cup \{x_9\} = \{x_5, x_8, x_9\}$ from Table 3.

By $x_9 \in E_4$ and $x_9 \in D'_1$, we calculate the approximations of D'_1 according to Proposition 3.2.1. Because $E_4 \not\subset \overline{R}^{II}_k(D_1)$ and $\overline{g}(E'_4, D'_1) = 1 \not> k$, there is $\overline{R}^{II}_k(D'_1) = \overline{R}^{II}_k(D_1)$ by the conclusion (2) of Proposition 3.2.1. Because $E_4 \not\subset \underline{R}^{II}_{(\alpha,\beta)}(D_1)$ and $P(D'_1|E'_4) = \frac{1}{3} \not\geq \alpha$, there is $\underline{R}^{II}_{(\alpha,\beta)}(D'_1) = \underline{R}^{II}_{(\alpha,\beta)}(D_1)$ by the conclusion (4) of Proposition 3.2.1.

By $x_9 \in E_4$ and $x_9 \notin D'_2$, we calculate the approximations of D'_2 according to Proposition 3.2.2. Because $E_4 \subseteq \overline{R}^{II}_k(D_2)$, there is $\overline{R}^{II}_k(D'_2) = \overline{R}^{II}_k(D_2) \cup \{x_9\}$ by the conclusion (1) of Proposition 3.2.2. Because $E_4 \subset \underline{R}^{II}_{(\alpha,\beta)}(D_2)$ and $P(D'_2|E'_4) = \frac{2}{3} \geq \alpha$, there is $\underline{R}^{II}_{(\alpha,\beta)}(D'_2) = \underline{R}^{II}_{(\alpha,\beta)}(D_2) \cup \{x_9\}$ by the conclusion (3) of Proposition 3.2.2.

3.3. The updating mechanisms for the concept approximations of Dq-DTRS with the batch insertion of objects

Based on the research of the first two Sections 3.1–3.2, we can achieve the approximation updating of DqI-DTRS and DqII-DTRS by iteration loop when many objects are inserted sequentially at time $t+1$. In this Section 3.3, we study the approximation updating methods for DqI-DTRS and DqII-DTRS when many objects are inserted in batches over time. Let ΔU denote the newly inserted object set at time $t+1$. Next, we directly analyze the effects of these new objects on the granular structures, then propose incremental approximation updating mechanisms of DqI-DTRS and DqII-DTRS.

Let $\Delta U/D = \{M_1, M_2, \ldots, M_s, M_{s+1}, \ldots, M_u\}$ and $\Delta U/C = \{N_1, N_2, \ldots, N_{s'}, N_{s'+1}, \ldots, N_v\}$ denote decision partition and conditional partition of ΔU, respectively. Then the decision partition of U' is $U'/D = \{D'_1, D'_2, \ldots, D'_s, D'_{s+1}, \ldots, D'_n, D'_{n+1}, \ldots, D'_{n+u-s}\}$, where for $i = 1, 2, \ldots, s$, $D'_i = D_i \cup M_i$ denote the original changed decision classes generated by incorporating the newly inserted decision classes into the corresponding original decision classes; for $i = s+1, s+2, \ldots, n$, $D'_i = D_i$ denote the original unchanged decision classes; for $i = n+1, n+2, \ldots, n+u-s$, $D'_i = M_{i-n+s}$ denote the newly inserted decision classes. The conditional partition of U' is $U'/C = \{E'_1, E'_2, \ldots, E'_{s'}, E'_{s'+1}, \ldots, E'_m, E'_{m+1}, \ldots, E'_{m+v-s'}\}$, where for $j = 1, 2, \ldots, s'$, $E'_j = E_j \cup N_j$ denote the original changed equivalence classes generated by incorporating the newly inserted equivalence classes into the corresponding original equivalence classes; for $j = s'+1, s'+2, \ldots, m$, $E'_j = E_j$ denote the original unchanged equivalence classes; for $j = m+1, m+2, \ldots, m+v-s'$, $E'_j = N_{j-m+s'}$ denote the newly inserted equivalence classes. In Table 4, we give information about decision classes, equivalence classes, conditional probability, external grade and internal grade when many objects are inserted at the same time.

Table 4
Information related to updating approximations with the batch insertion of objects.

Blocks		$D_i^{'}$	$E_j^{'}$	$P(D_i^{'}\|E_j^{'})$	Variation	$\underline{g}(E_j^{'}, D_i^{'})$	Variation	$\overline{g}(E_j^{'}, D_i^{'})$	Variation
1.	$i = 1, 2, \dots, s$ $j = 1, 2, \dots, s^{'}$	$D_i \cup M_i$	$E_j \cup N_j$	$\frac{\|E_j \cap D_i\| + \|N_j \cap M_i\|}{\|E_j\| + \|N_j\|}$	a	$\|E_j\| - \|E_j \cap D_i\|$ $+\|N_j\| - \|N_j \cap M_i\|$	$\geq$	$\|E_j \cap D_i\|$ $+\|N_j \cap M_i\|$	$\geq$
2.	$i = 1, 2, \dots, s$ $j = s^{'} + 1, \dots, m$	$D_i \cup M_i$	E_j	$\frac{\|E_j \cap D_i\|}{\|E_j\|}$	$=$	$\|E_j\| - \|E_j \cap D_i\|$	$=$	$\|E_j \cap D_i\|$	$=$
3.	$i = 1, 2, \dots, s$ $j = m + 1, \dots, m + v - s^{'}$	$D_i \cup M_i$	$N_{j-m+s^{'}}$	$\frac{\|N_{j-m+s^{'}} \cap M_i\|}{\|N_{j-m+s^{'}}\|}$	b	$\|N_{j-m+s^{'}}\| -$ $\|N_{j-m+s^{'}} \cap M_i\|$	b	$\|N_{j-m+s^{'}} \cap M_i\|$	b
4.	$i = s + 1, \dots, n$ $j = 1, 2, \dots, s^{'}$	D_i	$E_j \cup N_j$	$\frac{\|E_j \cap D_i\|}{\|E_j\| + \|N_j\|}$	$<$	$\|E_j\| + \|N_j\| -$ $\|E_j \cap D_i\|$	$>$	$\|E_j \cap D_i\|$	$=$
5.	$i = s + 1, \dots, n$ $j = s^{'} + 1, \dots, m$	D_i	E_j	$\frac{\|E_j \cap D_i\|}{\|E_j\|}$	$=$	$\|E_j\| - \|E_j \cap D_i\|$	$=$	$\|E_j \cap D_i\|$	$=$
6.	$i = s + 1, \dots, n$ $j = m + 1, \dots, m + v - s^{'}$	D_i	$N_{j-m+s^{'}}$	0	b	$\|N_{j-m+s^{'}}\|$	b	0	b
7.	$i = n + 1, \dots, n + u - s$ $j = 1, 2, \dots, s^{'}$	M_{i-n+s}	$E_j \cup N_j$	$\frac{\|N_j \cap M_{i-n+s}\|}{\|E_j\| + \|N_j\|}$	b	$\|E_j\| + \|N_j\| -$ $\|N_j \cap M_{i-n+s}\|$	b	$\|N_j \cap M_{i-n+s}\|$	b
8.	$i = n + 1, \dots, n + u - s$ $j = s^{'} + 1, \dots, m$	M_{i-n+s}	E_j	0	b	$\|E_j\|$	b	0	b
9.	$i = n + 1, \dots, n + u - s$ $j = m + 1, \dots, m + v - s^{'}$	M_{i-n+s}	$N_{j-m+s^{'}}$	$\frac{\|N_{j-m+s^{'}} \cap M_{i-n+s}\|}{\|N_{j-m+s^{'}}\|}$	b	$\|N_{j-m+s^{'}}\| -$ $\|N_{j-m+s^{'}} \cap M_{i-n+s}\|$	b	$\|N_{j-m+s^{'}} \cap M_{i-n+s}\|$	b

[a]Denotes that the relationship between the original probability and the corresponding current probability is uncertain.
[b]Denotes that the newly inserted information about conditional probability, external grade and internal grade at time $t + 1$.

From Table 4, we know that two approximations of the newly inserted decision classes ($D_i^{'}$, $i = n+1, \dots, n+u-s$) in DqI-DTRS and DqII-DTRS models can only be calculated by the formulas (2.1)–(2.4), because there is no prior knowledge from time t.

Proposition 3.3.1. *For the original changed decision classes $D_i^{'}$, $i = 1, 2, \dots, s$, the following properties for $\overline{R}^{I}_{(\alpha,\beta)}(D_i^{'})$ and $\underline{R}^{I}_{k}(D_i^{'})$ hold.*

(1) For $E_j, j = 1, 2, \dots, s'$,

(a_1) when $E_j \subseteq \overline{R}^{I}_{(\alpha,\beta)}(D_i)$, if $P(D_i^{'}|E_j^{'}) > \beta$, then $\overline{R}^{I}_{(\alpha,\beta)}(D_i^{'}) = \overline{R}^{I}_{(\alpha,\beta)}(D_i) \cup N_j$; otherwise, $\overline{R}^{I}_{(\alpha,\beta)}(D_i^{'}) = \overline{R}^{I}_{(\alpha,\beta)}(D_i) - E_j$.

(a_2) when $E_j \not\subset \overline{R}^{I}_{(\alpha,\beta)}(D_i)$, if $P(D_i^{'}|E_j^{'}) > \beta$, then $\overline{R}^{I}_{(\alpha,\beta)}(D_i^{'}) = \overline{R}^{I}_{(\alpha,\beta)}(D_i) \cup E_j^{'}$; otherwise, $\overline{R}^{I}_{(\alpha,\beta)}(D_i^{'}) = \overline{R}^{I}_{(\alpha,\beta)}(D_i)$.

(b_1) when $E_j \subseteq \underline{R}^{I}_{k}(D_i)$, if $\underline{g}(E_j^{'}, D_i^{'}) \leq k$, then $\underline{R}^{I}_{k}(D_i^{'}) = \underline{R}^{I}_{k}(D_i) \cup N_j$; otherwise, $\underline{R}^{I}_{k}(D_i^{'}) = \underline{R}^{I}_{k}(D_i) - E_j$.

(b_2) when $E_j \not\subset \underline{R}^{I}_{k}(D_i)$, there is $\underline{R}^{I}_{k}(D_i^{'}) = \underline{R}^{I}_{k}(D_i)$.

(2) For $E_j, j = s' + 1, s' + 2, \dots, m$, there are $\overline{R}^{I}_{(\alpha,\beta)}(D_i^{'}) = \overline{R}^{I}_{(\alpha,\beta)}(D_i)$ and $\underline{R}^{I}_{k}(D_i^{'}) = \underline{R}^{I}_{k}(D_i)$.

(3) For $E_j, j = m + 1, m + 2, \dots, m + v - s'$, there are $\overline{R}^{I}_{(\alpha,\beta)}(D_i^{'}) = \overline{R}^{I}_{(\alpha,\beta)}(D_i) \cup \{E_j^{'}|P(D_i^{'}|E_j^{'}) > \beta\}$ and $\underline{R}^{I}_{k}(D_i^{'}) = \underline{R}^{I}_{k}(D_i) \cup \{E_j^{'}|\underline{g}(E_j^{'}, D_i^{'}) \leq k\}$

Proposition 3.3.2. *For the original unchanged decision classes $D_i^{'}$, $i = s + 1, s + 2, \dots, n$, the following conclusions for $\overline{R}^{I}_{(\alpha,\beta)}(D_i^{'})$ and $\underline{R}^{I}_{k}(D_i^{'})$ hold.*

(1) For $E_j, j = 1, 2, \dots, s'$,

(a_1) when $E_j \subseteq \overline{R}^{I}_{(\alpha,\beta)}(D_i)$, if $P(D_i^{'}|E_j^{'}) > \beta$, then $\overline{R}^{I}_{(\alpha,\beta)}(D_i^{'}) = \overline{R}^{I}_{(\alpha,\beta)}(D_i) \cup E_j^{'} = \overline{R}^{I}_{(\alpha,\beta)}(D_i) \cup N_j$; otherwise, $\overline{R}^{I}_{(\alpha,\beta)}(D_i^{'}) = \overline{R}^{I}_{(\alpha,\beta)}(D_i) - E_j$.

(a_2) when $E_j \not\subset \overline{R}^{I}_{(\alpha,\beta)}(D_i)$, there is $\overline{R}^{I}_{(\alpha,\beta)}(D_i^{'}) = \overline{R}^{I}_{(\alpha,\beta)}(D_i)$.

(b_1) when $E_j \subseteq \underline{R}^{I}_{k}(D_i)$, if $\underline{g}(E_j^{'}, D_i^{'}) \leq k$, then $\underline{R}^{I}_{k}(D_i^{'}) = \underline{R}^{I}_{k}(D_i) \cup N_j$; otherwise, $\underline{R}^{I}_{k}(D_i^{'}) = \underline{R}^{I}_{k}(D_i) - E_j$.

(b_2) when $E_j \not\subset \underline{R}^{I}_{k}(D_i)$, there is $\underline{R}^{I}_{k}(D_i^{'}) = \underline{R}^{I}_{k}(D_i)$.

(2) For $E_j, j = s' + 1, s' + 2, \dots, m$, there are $\overline{R}^{I}_{(\alpha,\beta)}(D_i^{'}) = \overline{R}^{I}_{(\alpha,\beta)}(D_i)$ and $\underline{R}^{I}_{k}(D_i^{'}) = \underline{R}^{I}_{k}(D_i)$.

(3) For $E_j, j = m+1, m+2, \dots, m+v-s'$, there are $\overline{R}^{I}_{(\alpha,\beta)}(D_i^{'}) = \overline{R}^{I}_{(\alpha,\beta)}(D_i)$ and $\underline{R}^{I}_{k}(D_i^{'}) = \underline{R}^{I}_{k}(D_i) \cup \{N_{j-m+s'} \mid |N_{j-m+s'}| \leq k\}$

Table 5
The system with the insertion of objects.

U	a_1	a_2	a_3	a_4	d
x_1	1	0	1	1	0
x_2	1	1	1	1	0
x_3	1	0	1	1	1
x_4	0	1	0	1	1
x_5	1	1	1	0	1
x_6	1	0	1	1	0
x_7	1	1	1	1	0
x_8	1	1	1	0	1
$\mathbf{x_9}$	**1**	**0**	**1**	**1**	**0**
$\mathbf{x_{10}}$	**1**	**1**	**1**	**1**	**0**
$\mathbf{x_{11}}$	**0**	**0**	**1**	**1**	**2**
$\mathbf{x_{12}}$	**0**	**0**	**1**	**1**	**2**

According to the formulas (2.1)–(2.2) and the probability and external grade information of Table 4, Propositions 3.3.1–3.3.2 are valid.

Example 3. A decision information system $S = (U, A = C \cup D, V, f)$ at time t is shown in Table 2. At time $t+1$, the object set $\Delta U = \{x_9, x_{10}, x_{11}, x_{12}\}$ is inserted and its detailed information is shown in Table 5. We elaborate on the incremental approximation updating mechanisms of DqI-DTRS with the batch insertion of objects, where $\beta = \frac{1}{3}$ and $k = 1$.

From Table 5, there are $\Delta U/D = \{M_1 = \{x_9, x_{10}\}, M_2 = \{x_{11}, x_{12}\}\}$ and $\Delta U/C = \{N_1 = \{x_9\}, N_2 = \{x_{10}\}, N_3 = \{x_{11}, x_{12}\}\}$. Then we can get $U'/D = \{D_1^{'}, D_2^{'}, D_3^{'}\}$, where the original changed decision class $D_1^{'} = D_1 \cup M_1 = \{x_1, x_2, x_6, x_7, x_9, x_{10}\}$, the original unchanged decision class $D_2^{'} = D_2 = \{x_3, x_4, x_5, x_8\}$ and the newly inserted decision class $D_3^{'} = M_2 = \{x_{11}, x_{12}\}$. Similarly, we obtain $U'/C = \{E_1^{'}, E_2^{'}, E_3^{'}, E_4^{'}, E_5^{'}\}$, where the original changed equivalence classes $E_1' = E_1 \cup N_1 = \{x_1, x_3, x_6, x_9\}$ and $E_2' = E_2 \cup N_2 = \{x_2, x_7, x_{10}\}$, the original unchanged equivalence classes $E_3' = E_3 = \{x_4\}$ and $E_4' = E_4 = \{x_5, x_8\}$, the newly inserted equivalence class $E_5' = N_3 = \{x_{11}, x_{12}\}$. From Example 1, we know that $\overline{R}^{I}_{(\alpha,\beta)}(D_1) = E_1 \cup E_2$, $\underline{R}^{I}_{k}(D_1) = E_1 \cup E_2 \cup E_3$, $\overline{R}^{I}_{(\alpha,\beta)}(D_2) = E_3 \cup E_4$, $\underline{R}^{I}_{k}(D_2) = E_3 \cup E_4$.

First, we calculate the approximations of D_1' by Proposition 3.3.1. By $E_1 \subset \overline{R}^{I}_{(\alpha,\beta)}(D_1)$, $P(D_1'|E_1') = \frac{3}{4} > \beta$, $E_2 \subset \overline{R}^{I}_{(\alpha,\beta)}(D_1)$, $P(D_1'|E_2') = 1 > \beta$ and $P(D_1'|E_5') = 0 \not> \beta$, there is $\overline{R}^{I}_{(\alpha,\beta)}(D_1') = \overline{R}^{I}_{(\alpha,\beta)}(D_1) \cup N_1 \cup N_2$ from the conclusions (a_1) and (2–3) of Proposition 3.3.1. Similarly, by $E_1 \subset \underline{R}^{I}_{k}(D_1)$, $\underline{g}(E_1', D_1') = 1 \le k$, $E_2 \subset \underline{R}^{I}_{k}(D_1)$, $\underline{g}(E_2', D_1') = 0 \le k$ and $\underline{g}(E_5', D_1') = 2 \not\le k$, we can get $\underline{R}^{I}_{k}(D_1') = \underline{R}^{I}_{k}(D_1) \cup N_1 \cup N_2$ from the conclusions (b_1) and (2–3) of Proposition 3.3.1.

Then we calculate the approximations of D_2' by Proposition 3.3.2. By $E_1 \not\subset \overline{R}^{I}_{(\alpha,\beta)}(D_2)$, $E_2 \not\subset \overline{R}^{I}_{(\alpha,\beta)}(D_2)$, there is $\overline{R}^{I}_{(\alpha,\beta)}(D_2') = \overline{R}^{I}_{(\alpha,\beta)}(D_2)$ from the conclusions (a_2) and (2–3) of Proposition 3.3.2. Similarly, by $E_1 \not\subset \underline{R}^{I}_{k}(D_1)$, $E_2 \not\subset \underline{R}^{I}_{k}(D_1)$ and $|N_3| = 2 \not\le k$, we get $\underline{R}^{I}_{k}(D_2') = \underline{R}^{I}_{k}(D_2)$ from the conclusions (b_2) and (2–3) of Proposition 3.3.2.

Finally we calculate the approximations of D_3' according to formulas (2.1) and (2.2), which are $\overline{R}^{I}_{(\alpha,\beta)}(D_3') = E_5'$ and $\underline{R}^{I}_{k}(D_3') = E_3' \cup E_5'$.

Proposition 3.3.3. *For the original changed decision classes D_i', $i = 1, 2, \ldots, s$, the following properties for $\overline{R}^{II}_{k}(D_i')$ and $\underline{R}^{II}_{(\alpha,\beta)}(D_i')$ hold.*

(1) For $E_j, j = 1, 2, \ldots, s'$,

- *(a_1) when $E_j \subseteq \overline{R}^{II}_{k}(D_i)$, there is $\overline{R}^{II}_{k}(D_i') = \overline{R}^{II}_{k}(D_i) \cup E_j' = \overline{R}^{II}_{k}(D_i) \cup N_j$.*
- *(a_2) when $E_j \not\subset \overline{R}^{II}_{k}(D_i)$, if $\overline{g}(E_j', D_i') > k$, then $\overline{R}^{II}_{k}(D_i') = \overline{R}^{II}_{k}(D_i) \cup E_j'$; otherwise, $\overline{R}^{II}_{k}(D_i') = \overline{R}^{II}_{k}(D_i)$.*
- *(b_1) when $E_j \subseteq \underline{R}^{II}_{(\alpha,\beta)}(D_i)$, if $P(D_i'|E_j') \ge \alpha$, then $\underline{R}^{II}_{(\alpha,\beta)}(D_i') = \underline{R}^{II}_{(\alpha,\beta)}(D_i) \cup N_j$; otherwise, $\underline{R}^{II}_{(\alpha,\beta)}(D_i') = \underline{R}^{II}_{(\alpha,\beta)}(D_i) - E_j$.*
- *(b_2) when $E_j \not\subset \underline{R}^{II}_{(\alpha,\beta)}(D_i)$, if $P(D_i'|E_j') \ge \alpha$, then $\underline{R}^{II}_{(\alpha,\beta)}(D_i') = \underline{R}^{II}_{(\alpha,\beta)}(D_i) \cup E_j'$; otherwise, $\underline{R}^{II}_{(\alpha,\beta)}(D_i') = \underline{R}^{II}_{(\alpha,\beta)}(D_i)$.*

(2) For $E_j, j = s'+1, s'+2, \ldots, m$, there are $\overline{R}^{II}_{k}(D_i') = \overline{R}^{II}_{k}(D_i)$ and $\underline{R}^{II}_{(\alpha,\beta)}(D_i') = \underline{R}^{II}_{(\alpha,\beta)}(D_i)$.

(3) For $E_j, j = m+1, m+2, \ldots, m+v-s'$, there are $\overline{R}^{II}_{k}(D_i') = \overline{R}^{II}_{k}(D_i) \cup \{N_{j-m+s'} \mid \; |N_{j-m+s'} \cap M_i| > k\}$ and $\underline{R}^{II}_{(\alpha,\beta)}(D_i') = \underline{R}^{II}_{(\alpha,\beta)}(D_i) \cup \{N_{j-m+s'} \mid \; \frac{|N_{j-m+s'} \cap M_i|}{|N_{j-m+s'}|} \ge \alpha\}$.

Proposition 3.3.4. *For the original changed decision classes D_i', $i = s+1, s+2, \ldots, n$, the following properties for $\overline{R}^{II}_{k}(D_i')$ and $\underline{R}^{II}_{(\alpha,\beta)}(D_i')$ hold.*

(1) For $E_j, j = 1, 2, \ldots, s'$,

- *(a) if $E_j \subseteq \overline{R}^{II}_{k}(D_i)$, then $\overline{R}^{II}_{k}(D_i') = \overline{R}^{II}_{k}(D_i) \cup N_j$; otherwise, $\overline{R}^{II}_{k}(D_i') = \overline{R}^{II}_{k}(D_i)$.*
- *(b_1) when $E_j \subseteq \underline{R}^{II}_{(\alpha,\beta)}(D_i)$, if $P(D_i'|E_j') \ge \alpha$, then $\underline{R}^{II}_{(\alpha,\beta)}(D_i') = \underline{R}^{II}_{(\alpha,\beta)}(D_i) \cup N_j$; otherwise, $\underline{R}^{II}_{(\alpha,\beta)}(D_i') = \underline{R}^{II}_{(\alpha,\beta)}(D_i) - E_j$.*
- *(b_2) when $E_j \not\subset \underline{R}^{II}_{(\alpha,\beta)}(D_i)$, there is $\underline{R}^{II}_{(\alpha,\beta)}(D_i') = \underline{R}^{II}_{(\alpha,\beta)}(D_i)$.*

(2) For $E_j, j = s'+1, s'+2, \ldots, m+v-s'$, there are $\overline{R}^{II}_{k}(D_i') = \overline{R}^{II}_{k}(D_i)$ and $\underline{R}^{II}_{(\alpha,\beta)}(D_i') = \underline{R}^{II}_{(\alpha,\beta)}(D_i)$.

According to the formulas (2.3)–(2.4) and the probability and internal grade information of Table 4, Propositions 3.3.3–3.3.4 are obvious.

Example 4 (*Continuation of Example 3*)**.** We elaborate on the incremental approximation updating mechanisms of DqII-DTRS with the batch insertion of objects, where $k = 1$ and $\alpha = \frac{2}{3}$. From Example 2, we know that $\overline{R}^{II}_{k}(D_1) = E_1 \cup E_2$, $\underline{R}^{II}_{(\alpha,\beta)}(D_1) = E_1 \cup E_2$, $\overline{R}^{II}_{k}(D_2) = E_4$, $\underline{R}^{II}_{(\alpha,\beta)}(D_2) = E_3 \cup E_4$.

First, we calculate the approximations of D_1' by Proposition 3.3.3. By $E_1 \subset \overline{R}^{II}_{k}(D_1)$, $E_2 \subset \overline{R}^{II}_{k}(D_1)$ and $|N_3 \cap M_1| = 0 \not> k$, there is $\overline{R}^{II}_{k}(D_1') = \overline{R}^{II}_{k}(D_1) \cup N_1 \cup N_2$ from the conclusions (a_1) and (2–3) of Proposition 3.3.3. By $E_1 \subset \underline{R}^{II}_{(\alpha,\beta)}(D_1)$, $P(D_1'|E_1') = \frac{3}{4} \ge \alpha$, $E_2 \subset \underline{R}^{II}_{(\alpha,\beta)}(D_1)$, $P(D_1'|E_2') = 1 \ge \alpha$ and $\frac{|N_3 \cap M_1|}{|N_3|} = 0 \not\ge \alpha$, there is $\underline{R}^{II}_{(\alpha,\beta)}(D_1') = \underline{R}^{II}_{(\alpha,\beta)}(D_1) \cup N_1 \cup N_2$ by the conclusions (b_1) and (2–3) of Proposition 3.3.3.

Then we calculate the approximations of D_2' by Proposition 3.3.4. By $E_1 \not\subset \overline{R}^{II}_{k}(D_2)$ and $E_2 \not\subset \overline{R}^{II}_{k}(D_2)$, there is $\overline{R}^{II}_{k}(D_2') = \overline{R}^{II}_{k}(D_2)$ from the conclusions (a) and (2) of Proposition 3.3.4. By $E_1 \not\subset \underline{R}^{II}_{(\alpha,\beta)}(D_2)$ and $E_2 \not\subset \underline{R}^{II}_{(\alpha,\beta)}(D_2)$, there is $\underline{R}^{II}_{(\alpha,\beta)}(D_2') = \underline{R}^{II}_{(\alpha,\beta)}(D_2)$ from the conclusions (b_2) and (2) of Proposition 3.3.4.

Finally we calculate the approximations of D_3' according to formulas (2.3) and (2.4), which are $\overline{R}^{II}_{k}(D_3') = E_5'$ and $\underline{R}^{II}_{(\alpha,\beta)}(D_3') = E_5'$.

4. Incremental approximation updating of Dq-DTRS with the sequential and batch deletion of objects

In this section, we first study the incremental approximation updating mechanisms of DqI-DTRS and DqII-DTRS with the sequential deletion of objects over time. We mainly present the approximation updating process when deleting an object, and that process of the sequential deletion of many objects is realized by iteration.

Let x^- be a deleted object, where $x^- \in D_i$ and $x^- \in E_j$. So we have $D_i' = D_i - \{x^-\}$ and $E_j' = E_j - \{x^-\}$. The other equivalence classes and decision classes that do not contain x^- remain unchanged.

According to the information in Table 1, if the decision class including the object x^- is a singleton set, then $n' = n - 1$; otherwise, $n' = n$. If the equivalence class including the object x^- is a singleton set, then $m' = m - 1$; otherwise, $m' = m$. Detailed variations of conditional probability, external grade and internal grade are shown in the following lemma, where the other decision classes, equivalence classes, conditional probability, external grade and internal grade remain unchanged except that mentioned in the conclusions of Lemma 4.1.

Lemma 4.1. *When an object x^- is deleted at time $t+1$, we have the following conclusions:*

(1) If $|D_i| > 1$ ($i \in \{1, 2, \ldots, n\}$) and $|E_j| > 1$ ($j \in \{1, 2, \ldots, m\}$), there are
$p_{1j}' > p_{1j}, p_{2j}' > p_{2j}, \ldots, p_{i-1,j}' > p_{i-1,j}, p_{ij}' < p_{ij}, p_{i+1,j}' > p_{i+1,j}, \ldots, p_{nj}' > p_{nj}$;
$e_{1j}' < e_{1j}, e_{2j}' < e_{2j}, \ldots, e_{i-1,j}' < e_{i-1,j}, e_{i+1,j}' < e_{i+1,j}, \ldots, e_{nj}' < e_{nj}$;
$i_{ij}' < i_{ij}$.

(2) If $|D_i| > 1$ ($i \in \{1, 2, \ldots, n\}$) and $|E_j| = 1$ ($j \in \{1, 2, \ldots, m\}$), there are
$p_{1,j}' = p_{2,j}' = \cdots = p_{n,j}' = \times$; $e_{1,j}' = e_{2,j}' = \cdots = e_{n,j}' = \times$;

$i'_{1,j} = i'_{2,j} = \cdots = i'_{n,j} = \times$; where $\times$ represents the deleted value.

(3) *If $|D_i| = 1$ ($i \in \{1, 2, \ldots, n\}$) and $|E_j| > 1$ ($j \in \{1, 2, \ldots, m\}$), there are*
$p'_{i,1} = p'_{i,2} = \cdots = p'_{i,m} = \times$, $p'_{1j} > p_{1j}, p'_{2j} > p_{2j}, \ldots, p'_{i-1,j} > p_{i-1,j}, p'_{i+1,j} > p_{i+1,j}, \ldots, p'_{nj} > p_{nj}$;
$e'_{i,1} = e'_{i,2} = \cdots = e'_{i,m} = \times$, $e'_{1j} < e_{1j}, e'_{2j} < e_{2j}, \ldots, e'_{i-1,j} < e_{i-1,j}, e'_{i+1,j} < e_{i+1,j}, \ldots, e'_{nj} < e_{nj}$;
$i'_{i,1} = i'_{i,2} = \cdots = i'_{i,m} = \times$.

(4) *If $|D_i| = 1$ ($i \in \{1, 2, \ldots, n\}$) and $|E_j| = 1$ ($j \in \{1, 2, \ldots, m\}$), there are*
$p'_{i,1} = p'_{i,2} = \cdots = p'_{i,m} = \times$, $p'_{1,j} = p'_{2,j} = \cdots = p'_{n,j} = \times$;
$e'_{i,1} = e'_{i,2} = \cdots = e'_{i,m} = \times$, $e'_{1,j} = e'_{2,j} = \cdots = e'_{n,j} = \times$;
$i'_{i,1} = i'_{i,2} = \cdots = i'_{i,m} = \times$, $i'_{1,j} = i'_{2,j} = \cdots = i'_{n,j} = \times$.

The conclusions of Lemma 4.1 can be obtained similar to the proving method of Lemma 3.1. In the following, we give more intuitive results of the changes of conditional probability, external grade and internal grade with the deletion of an object.

$$P'_{(1)} = \begin{pmatrix} p_{11} & \cdots & p_{1,j-1} & > p_{1j} & p_{1,j+1} & \cdots & p_{1m} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ p_{i-1,1} & \cdots & p_{i-1,j-1} & > p_{i-1,j} & p_{i-1,j+1} & \cdots & p_{i-1,m} \\ p_{i1} & \cdots & p_{i,j-1} & < p_{ij} & p_{i,j+1} & \cdots & p_{im} \\ p_{i+1,1} & \cdots & p_{i+1,j-1} & > p_{i+1j} & p_{i+1,j+1} & \cdots & p_{i+1,m} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ p_{n1} & \cdots & p_{n,j-1} & > p_{nj} & p_{n,j+1} & \cdots & p_{nm} \end{pmatrix}$$

$$P'_{(2)} = \begin{pmatrix} p_{11} & \cdots & p_{1,j-1} & \times & p_{1,j+1} & \cdots & p_{1m} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ p_{i-1,1} & \cdots & p_{i-1,j-1} & \times & p_{i-1,j+1} & \cdots & p_{i-1,m} \\ p_{i1} & \cdots & p_{i,j-1} & \times & p_{i,j+1} & \cdots & p_{im} \\ p_{i+1,1} & \cdots & p_{i+1,j-1} & \times & p_{i+1,j+1} & \cdots & p_{i+1,m} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ p_{n1} & \cdots & p_{n,j-1} & \times & p_{n,j+1} & \cdots & p_{nm} \end{pmatrix}$$

$$E'_{(1)} = \begin{pmatrix} e_{11} & \cdots & e_{1,j-1} & < e_{1j} & e_{1,j+1} & \cdots & e_{1m} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ e_{i-1,1} & \cdots & e_{i-1,j-1} & < e_{i-1,j} & e_{i-1,j+1} & \cdots & e_{i-1,m} \\ e_{i1} & \cdots & e_{i,j-1} & e_{ij} & e_{i,j+1} & \cdots & e_{im} \\ e_{i+1,1} & \cdots & e_{i+1,j-1} & < e_{i+1j} & e_{i+1,j+1} & \cdots & e_{i+1,m} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ e_{n1} & \cdots & e_{n,j-1} & < e_{nj} & e_{n,j+1} & \cdots & e_{nm} \end{pmatrix}$$

$$E'_{(2)} = \begin{pmatrix} e_{11} & \cdots & e_{1,j-1} & \times & e_{1,j+1} & \cdots & e_{1m} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ e_{i-1,1} & \cdots & e_{i-1,j-1} & \times & e_{i-1,j+1} & \cdots & e_{i-1,m} \\ e_{i1} & \cdots & e_{i,j-1} & \times & e_{i,j+1} & \cdots & e_{im} \\ e_{i+1,1} & \cdots & e_{i+1,j-1} & \times & e_{i+1,j+1} & \cdots & e_{i+1,m} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ e_{n1} & \cdots & e_{n,j-1} & \times & e_{n,j+1} & \cdots & e_{nm} \end{pmatrix}$$

$$I'_{(1)} = \begin{pmatrix} i_{11} & \cdots & i_{1,j-1} & i_{1j} & i_{1,j+1} & \cdots & i_{1m} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ i_{i-1,1} & \cdots & i_{i-1,j-1} & i_{i-1,j} & i_{i-1,j+1} & \cdots & i_{i-1,m} \\ i_{i1} & \cdots & i_{i,j-1} & < i_{ij} & i_{i,j+1} & \cdots & i_{im} \\ i_{i+1,1} & \cdots & i_{i+1,j-1} & i_{i+1,j} & i_{i+1,j+1} & \cdots & i_{i+1,m} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ i_{n1} & \cdots & i_{n,j-1} & i_{nj} & i_{n,j+1} & \cdots & i_{nm} \end{pmatrix}$$

$$I'_{(2)} = \begin{pmatrix} i_{11} & \cdots & i_{1,j-1} & \times & i_{1,j+1} & \cdots & i_{1m} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ i_{i-1,1} & \cdots & i_{i-1,j-1} & \times & i_{i-1,j+1} & \cdots & i_{i-1,m} \\ i_{i1} & \cdots & i_{i,j-1} & \times & i_{i,j+1} & \cdots & i_{im} \\ i_{i+1,1} & \cdots & i_{i+1,j-1} & \times & i_{i+1,j+1} & \cdots & i_{i+1,m} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ i_{n1} & \cdots & i_{n,j-1} & \times & i_{n,j+1} & \cdots & i_{nm} \end{pmatrix}$$

$$P'_{(3)} = \begin{pmatrix} p_{11} & \cdots & p_{1,j-1} & > p_{1j} & p_{1,j+1} & \cdots & p_{1m} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ p_{i-1,1} & \cdots & p_{i-1,j-1} & > p_{i-1,j} & p_{i-1,j+1} & \cdots & p_{i-1,m} \\ \times & \cdots & \times & \times & \times & \cdots & \times \\ p_{i+1,1} & \cdots & p_{i+1,j-1} & > p_{i+1,j} & p_{i+1,j+1} & \cdots & p_{i+1,m} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ p_{n1} & \cdots & p_{n,j-1} & > p_{nj} & p_{n,j+1} & \cdots & p_{nm} \end{pmatrix}$$

$$P'_{(4)} = \begin{pmatrix} p_{11} & \cdots & p_{1,j-1} & \times & p_{1,j+1} & \cdots & p_{1m} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ p_{i-1,1} & \cdots & p_{i-1,j-1} & \times & p_{i-1,j+1} & \cdots & p_{i-1,m} \\ \times & \cdots & \times & \times & \times & \cdots & \times \\ p_{i+1,1} & \cdots & p_{i+1,j-1} & \times & p_{i+1,j+1} & \cdots & p_{i+1,m} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ p_{n1} & \cdots & p_{n,j-1} & \times & p_{n,j+1} & \cdots & p_{nm} \end{pmatrix}$$

$$E'_{(3)} = \begin{pmatrix} e_{11} & \cdots & e_{1,j-1} & < e_{1j} & e_{1,j+1} & \cdots & e_{1m} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ e_{i-1,1} & \cdots & e_{i-1,j-1} & < e_{i-1,j} & e_{i-1,j+1} & \cdots & e_{i-1,m} \\ \times & \cdots & \times & \times & \times & \cdots & \times \\ e_{i+1,1} & \cdots & e_{i+1,j-1} & < e_{i+1,j} & e_{i+1,j+1} & \cdots & e_{i+1,m} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ e_{n1} & \cdots & e_{n,j-1} & < e_{nj} & e_{n,j+1} & \cdots & e_{nm} \end{pmatrix}$$

$$E'_{(4)} = \begin{pmatrix} e_{11} & \cdots & e_{1,j-1} & \times & e_{1,j+1} & \cdots & e_{1m} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ e_{i-1,1} & \cdots & e_{i-1,j-1} & \times & e_{i-1,j+1} & \cdots & e_{i-1,m} \\ \times & \cdots & \times & \times & \times & \cdots & \times \\ e_{i+1,1} & \cdots & e_{i+1,j-1} & \times & e_{i+1,j+1} & \cdots & e_{i+1,m} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ e_{n1} & \cdots & e_{n,j-1} & \times & e_{n,j+1} & \cdots & e_{nm} \end{pmatrix}$$

$$I'_{(3)} = \begin{pmatrix} i_{11} & \cdots & i_{1,j-1} & i_{1j} & i_{1,j+1} & \cdots & i_{1m} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ i_{i-1,1} & \cdots & i_{i-1,j-1} & i_{i-1,j} & i_{i-1,j+1} & \cdots & i_{i-1,m} \\ \times & \cdots & \times & \times & \times & \cdots & \times \\ i_{i+1,1} & \cdots & i_{i+1,j-1} & i_{i+1,j} & i_{i+1,j+1} & \cdots & i_{i+1,m} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ i_{n1} & \cdots & i_{n,j-1} & i_{nj} & i_{n,j+1} & \cdots & i_{nm} \end{pmatrix}$$

$$I'_{(4)} = \begin{pmatrix} i_{11} & \cdots & i_{1,j-1} & \times & i_{1,j+1} & \cdots & i_{1m} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ i_{i-1,1} & \cdots & i_{i-1,j-1} & \times & i_{i-1,j+1} & \cdots & i_{i-1,m} \\ \times & \cdots & \times & \times & \times & \cdots & \times \\ i_{i+1,1} & \cdots & i_{i+1,j-1} & \times & i_{i+1,j+1} & \cdots & i_{i+1,m} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ i_{n1} & \cdots & i_{n,j-1} & \times & i_{n,j+1} & \cdots & i_{nm} \end{pmatrix}$$

In the above $P'_{(1)-(4)}, E'_{(1)-(4)}, I'_{(1)-(4)}$, the blue crosses represent the deleted values. From the $P'_{(1)-(4)}, E'_{(1)-(4)}, I'_{(1)-(4)}$, we can see the information association and the change between time t and time $t+1$ with the deletion of an object. When updating the upper and lower approximations with the deletion of an object, we are mainly concerned with the information changes between the equivalence class including this object and decision classes, that is, the jth column of the above matrices.

4.1. The updating mechanisms for the concept approximations of DqI-DTRS with the deletion of an object

We first study the updating mechanisms for the approximations of DqI-DTRS when x^- is deleted from the original decision system at time $t+1$. Let $x^- \in E_j$, then $E'_j = E_j - \{x^-\}$.

Proposition 4.1.1. *Let $x^- \in E_j$, if $x^- \in D_i$, the following properties for $\overline{R}^I_{(\alpha,\beta)}(D'_i)$ and $\underline{R}^I_k(D'_i)$ hold.*

(1) *When $E_j \subseteq \overline{R}^I_{(\alpha,\beta)}(D_i)$, if $P(D'_i|E'_j) > \beta$, then $\overline{R}^I_{(\alpha,\beta)}(D'_i) = \overline{R}^I_{(\alpha,\beta)}(D_i) - \{x^-\}$; otherwise, $\overline{R}^I_{(\alpha,\beta)}(D'_i) = \overline{R}^I_{(\alpha,\beta)}(D_i) - E_j$.*
(2) *When $E_j \not\subset \overline{R}^I_{(\alpha,\beta)}(D_i)$, there is $\overline{R}^I_{(\alpha,\beta)}(D'_i) = \overline{R}^I_{(\alpha,\beta)}(D_i)$.*
(3) *When $E_j \subseteq \underline{R}^I_k(D_i)$, there is $\underline{R}^I_k(D'_i) = \underline{R}^I_k(D_i) - \{x^-\}$.*
(4) *When $E_j \not\subset \underline{R}^I_k(D_i)$, there is $\underline{R}^I_k(D'_i) = \underline{R}^I_k(D_i)$.*

Proof. When $x^- \in E_j$ and $x^- \in D_i$, we can get $E'_j = E_j - \{x^-\}$, $D'_i = D_i - \{x^-\}$ and $|E'_j \cap D'_i| = |E_j \cap D_i| - 1$.

(1) By $E_j \subseteq \overline{R}^I_{(\alpha,\beta)}(D_i)$ and $P(D'_i|E'_j) > \beta$, there is $\overline{R}^I_{(\alpha,\beta)}(D'_i) = \overline{R}^I_{(\alpha,\beta)}(D_i) - \{x^-\}$. On the contrary, according to $E_j \subseteq \overline{R}^I_{(\alpha,\beta)}(D_i)$ and $E'_j \not\subset \overline{R}^I_{(\alpha,\beta)}(D'_i)$, we can get $\overline{R}^I_{(\alpha,\beta)}(D'_i) = \overline{R}^I_{(\alpha,\beta)}(D_i) - E_j$.

(2) According to $E_j \not\subset \overline{R}^{I}_{(\alpha,\beta)}(D_i)$ and $P(D_i'|E_j') = \frac{|E_j \cap D_i|-1}{|E_j|-1} < P(D_i|E_j)$, we can obtain that $E_j' \not\subset \overline{R}^{I}_{(\alpha,\beta)}(D_i')$. So $\overline{R}^{I}_{(\alpha,\beta)}(D_i') = \overline{R}^{I}_{(\alpha,\beta)}(D_i)$.

(3) By $E_j \subseteq \underline{R}^{I}_{k}(D_i)$ and $\underline{g}(E_j', D_i') = \underline{g}(E_j, D_i)$, there is $E_j' \subseteq \underline{R}^{I}_{k}(D_i')$. Therefore, $\underline{R}^{I}_{k}(D_i') = \underline{R}^{I}_{k}(D_i) - \{x^-\}$.

(4) According to $E_j \not\subset \underline{R}^{I}_{k}(D_i)$ and $\underline{g}(E_j', D_i') = \underline{g}(E_j, D_i)$, we can get $\underline{R}^{I}_{k}(D_i') = \underline{R}^{I}_{k}(D_i)$. □

From Proposition 4.1.1, we first consider a prerequisite, namely the deleted object belongs to one original decision class. In the DqI-DTRS model, when the original equivalence class to which the deleted object belongs is contained in the upper (lower) approximation of the original decision class, the upper (lower) approximation of the corresponding current decision class needs to be updated.

Proposition 4.1.2. *Let $x^- \in E_j$, If $x^- \notin D_i$, the following properties for $\overline{R}^{I}_{(\alpha,\beta)}(D_i')$ and $\underline{R}^{I}_{k}(D_i')$ hold.*

(1) When $E_j \subseteq \overline{R}^{I}_{(\alpha,\beta)}(D_i)$, there is $\overline{R}^{I}_{(\alpha,\beta)}(D_i') = \overline{R}^{I}_{(\alpha,\beta)}(D_i) - \{x^-\}$.
(2) When $E_j \not\subset \overline{R}^{I}_{(\alpha,\beta)}(D_i)$, if $P(D_i'|E_j') > \beta$, then $\overline{R}^{I}_{(\alpha,\beta)}(D_i') = \overline{R}^{I}_{(\alpha,\beta)}(D_i) \cup E_j'$; otherwise, $\overline{R}^{I}_{(\alpha,\beta)}(D_i') = \overline{R}^{I}_{(\alpha,\beta)}(D_i)$.
(3) When $E_j \subseteq \underline{R}^{I}_{k}(D_i)$, there is $\underline{R}^{I}_{k}(D_i') = \underline{R}^{I}_{k}(D_i) - \{x^-\}$.
(4) When $E_j \not\subset \underline{R}^{I}_{k}(D_i)$, if $\underline{g}(E_j', D_i') \le k$, then $\underline{R}^{I}_{k}(D_i') = \underline{R}^{I}_{k}(D_i) \cup E_j'$; otherwise, $\underline{R}^{I}_{k}(D_i') = \underline{R}^{I}_{k}(D_i)$.

Proof. From $x^- \in E_j$ and $x^- \notin D_i$, we can get $E_j' = E_j - \{x^-\}$, $D_i' = D_i$ and $|E_j' \cap D_i'| = |E_j \cap D_i|$. Furthermore, two inequations $P(D_i'|E_j') > P(D_i|E_j)$ and $\underline{g}(E_j', D_i') < \underline{g}(E_j, D_i)$ are obtained. The properties of $\overline{R}^{I}_{(\alpha,\beta)}(D_i')$ and $\underline{R}^{I}_{k}(D_i')$ can be obtained by analogous to the proof method of Proposition 4.1.1. □

In Proposition 4.1.2, the prerequisite is that the deleted object does not belong to one original decision class. In the DqI-DTRS model, when the original equivalence class to which the deleted object belongs is contained in the upper (lower) approximation of the original decision class, the upper (lower) approximation of the corresponding current decision class can be updated directly. When this original equivalence class is not contained in the upper (lower) approximation of the original decision class, we need to further determine whether the current conditional probability (external grade) meets the threshold requirement. Then we decide whether to update the upper (lower) approximation.

Example 5. A decision information system $S = (U, A = C \cup D, V, f)$ at time t is shown in Table 2. At time $t+1$, the object x_8 is deleted. We elaborate on the incremental approximation updating mechanisms of DqI-DTRS with the deletion of an object, where $\beta = \frac{1}{3}$ and $k = 1$. From Example 1, we know that $\overline{R}^{I}_{(\alpha,\beta)}(D_1) = E_1 \cup E_2$, $\underline{R}^{I}_{k}(D_1) = E_1 \cup E_2 \cup E_3$, $\overline{R}^{I}_{(\alpha,\beta)}(D_2) = E_3 \cup E_4$, $\underline{R}^{I}_{k}(D_2) = E_3 \cup E_4$. At time $t+1$, we know that $D_1' = D_1 = \{x_1, x_2, x_6, x_7\}$, $D_2' = D_2 - \{x_8\} = \{x_3, x_4, x_5\}$ and $E_1' = E_1 = \{x_1, x_3, x_6\}$, $E_2' = E_2 = \{x_2, x_7\}$, $E_3' = E_3 = \{x_4\}$, $E_4' = E_4 - \{x_8\} = \{x_5\}$ from Table 2.

By $x_8 \in E_4$ and $x_8 \in D_2$, we calculate the approximations of D_2' according to Proposition 4.1.1. Because $E_4 \subset \overline{R}^{I}_{(\alpha,\beta)}(D_2)$, $P(D_2'|E_4') = 1 > \beta$, $E_4 \subset \underline{R}^{I}_{k}(D_2)$, we get $\overline{R}^{I}_{(\alpha,\beta)}(D_2') = \overline{R}^{I}_{(\alpha,\beta)}(D_2) - \{x_8\}$ and $\underline{R}^{I}_{k}(D_2') = \underline{R}^{I}_{k}(D_2) - \{x_8\}$ from the conclusions (1) and (3) of Proposition 4.1.1.

By $x_8 \in E_4$ and $x_8 \notin D_1$, we calculate the approximations of D_1' according to Proposition 4.1.2. Because $E_4 \not\subset \overline{R}^{I}_{(\alpha,\beta)}(D_1)$, $P(D_1'|E_4') = 0 \not> \beta$, $E_4 \not\subset \underline{R}^{I}_{k}(D_1)$ and $\underline{g}(E_4', D_1') = 1 \le k$, we get $\overline{R}^{I}_{(\alpha,\beta)}(D_1') = \overline{R}^{I}_{(\alpha,\beta)}(D_1)$ and $\underline{R}^{I}_{k}(D_1') = \underline{R}^{I}_{k}(D_1) \cup E_4'$ from the conclusions (2) and (4) of Proposition 4.1.2.

4.2. The updating mechanisms for the concept approximations of DqII-DTRS with the deletion of an object

In the following, we propose the updating mechanisms for the approximations of DqII-DTRS when an object x^- is deleted. Let $x^- \in E_j$, then $E_j' = E_j - \{x^-\}$.

Proposition 4.2.1. *Let $x^- \in E_j$, if $x^- \in D_i$, the following properties for $\overline{R}^{II}_{k}(D_i')$ and $\underline{R}^{II}_{(\alpha,\beta)}(D_i')$ hold.*

(1) When $E_j \subseteq \overline{R}^{II}_{k}(D_i)$, if $\overline{g}(E_j', D_i') > k$, then $\overline{R}^{II}_{k}(D_i') = \overline{R}^{II}_{k}(D_i) - \{x^-\}$; otherwise, $\overline{R}^{II}_{k}(D_i') = \overline{R}^{II}_{k}(D_i) - E_j$.
(2) When $E_j \not\subset \overline{R}^{II}_{k}(D_i)$, there is $\overline{R}^{II}_{k}(D_i') = \overline{R}^{II}_{k}(D_i)$.
(3) When $E_j \subseteq \underline{R}^{II}_{(\alpha,\beta)}(D_i)$, if $P(D_i'|E_j') \ge \alpha$, then $\underline{R}^{II}_{(\alpha,\beta)}(D_i') = \underline{R}^{II}_{(\alpha,\beta)}(D_i) - \{x^-\}$; otherwise, $\underline{R}^{II}_{(\alpha,\beta)}(D_i') = \underline{R}^{II}_{(\alpha,\beta)}(D_i) - E_j$.
(4) When $E_j \not\subset \underline{R}^{II}_{(\alpha,\beta)}(D_i)$, there is $\underline{R}^{II}_{(\alpha,\beta)}(D_i') = \underline{R}^{II}_{(\alpha,\beta)}(D_i)$.

Proof. When $x^- \in E_j$ and $x^- \in D_i$, we can get $E_j' = E_j - \{x^-\}$, $D_i' = D_i - \{x^-\}$ and $|E_j' \cap D_i'| = |E_j \cap D_i| - 1$. Furthermore, two inequations $\overline{g}(E_j', D_i') < \overline{g}(E_j, D_i)$ and $P(D_i'|E_j') < P(D_i|E_j)$ are obtained.

(1) According to $E_j \subseteq \overline{R}^{II}_{k}(D_i)$ and $\overline{g}(E_j', D_i') > k$, we can get $\overline{R}^{II}_{k}(D_i') = \overline{R}^{II}_{k}(D_i) - \{x^-\}$. Otherwise, by $E_j \subseteq \overline{R}^{II}_{k}(D_i)$ and $E_j' \not\subset \overline{R}^{II}_{k}(D_i')$, there is $\overline{R}^{II}_{k}(D_i') = \overline{R}^{II}_{k}(D_i) - E_j$.

(2) By $E_j \not\subset \overline{R}^{II}_{k}(D_i)$, there is $E_j' \not\subset \overline{R}^{II}_{k}(D_i')$. Therefore, $\overline{R}^{II}_{k}(D_i') = \overline{R}^{II}_{k}(D_i)$.

(3) When $E_j \subseteq \underline{R}^{II}_{(\alpha,\beta)}(D_i)$, if $P(D_i'|E_j') \ge \alpha$, there is $E_j' \subseteq \underline{R}^{II}_{(\alpha,\beta)}(D_i')$. Furthermore, we can get $\underline{R}^{II}_{(\alpha,\beta)}(D_i') = \underline{R}^{II}_{(\alpha,\beta)}(D_i) - \{x^-\}$. On the contrary, by $E_j' \not\subset \underline{R}^{II}_{(\alpha,\beta)}(D_i')$ and $E_j \subseteq \underline{R}^{II}_{(\alpha,\beta)}(D_i)$, we can obtain $\underline{R}^{II}_{(\alpha,\beta)}(D_i') = \underline{R}^{II}_{(\alpha,\beta)}(D_i) - E_j$.

(4) By $E_j \not\subset \underline{R}^{II}_{(\alpha,\beta)}(D_i)$ and $P(D_i'|E_j') < P(D_i|E_j)$, it is true that $\underline{R}^{II}_{(\alpha,\beta)}(D_i') = \underline{R}^{II}_{(\alpha,\beta)}(D_i)$. □

From Proposition 4.2.1, we first consider a prerequisite, namely the deleted object belongs to one original decision class. In the DqII-DTRS model, when the original equivalence class to which the deleted object belongs is contained in the upper (lower) approximation of the original decision class, the upper (lower) approximation of the corresponding current decision class needs to be updated.

Proposition 4.2.2. *Let $x^- \in E_j$, if $x^- \notin D_i$, the following properties for $\overline{R}^{II}_{k}(D_i')$ and $\underline{R}^{II}_{(\alpha,\beta)}(D_i')$ hold.*

(1) When $E_j \subseteq \overline{R}^{II}_{k}(D_i)$, there is $\overline{R}^{II}_{k}(D_i') = \overline{R}^{II}_{k}(D_i) - \{x^-\}$.
(2) When $E_j \not\subset \overline{R}^{II}_{k}(D_i)$, there is $\overline{R}^{II}_{k}(D_i') = \overline{R}^{II}_{k}(D_i)$.
(3) When $E_j \subseteq \underline{R}^{II}_{(\alpha,\beta)}(D_i)$, there is $\underline{R}^{II}_{(\alpha,\beta)}(D_i') = \underline{R}^{II}_{(\alpha,\beta)}(D_i) - \{x^-\}$.
(4) When $E_j \not\subset \underline{R}^{II}_{(\alpha,\beta)}(D_i)$, if $P(D_i'|E_j') \ge \alpha$, then $\underline{R}^{II}_{(\alpha,\beta)}(D_i') = \underline{R}^{II}_{(\alpha,\beta)}(D_i) \cup E_j'$; otherwise, $\underline{R}^{II}_{(\alpha,\beta)}(D_i') = \underline{R}^{II}_{(\alpha,\beta)}(D_i)$.

Proof. The properties about $\overline{R}^{II}_{k}(D_i')$ and $\underline{R}^{II}_{(\alpha,\beta)}(D_i')$ can be easily obtained by formulas (2.3)–(2.4) and two expressions $\overline{g}(E_j', D_i') = \overline{g}(E_j, D_i)$ and $P(D_i'|E_j') > P(D_i|E_j)$. □

Table 6
Information related to updating approximations with the batch deletion of objects.

Blocks		$D_i^{'}$	$E_j^{'}$	$P(D_i^{'}\|E_j^{'})$	Variation	$\underline{g}(E_j^{'}, D_i^{'})$	Variation	$\overline{g}(E_j^{'}, D_i^{'})$	Variation
1.	$i = 1, 2, \ldots, s$ $j = 1, 2, \ldots, s^{'}$	$D_i - M_i$	$E_j - N_j$	$\frac{\|E_j \cap D_i\| - \|N_j \cap M_i\|}{\|E_j\| - \|N_j\|}$	[a]	$\|E_j\| - \|E_j \cap D_i\|$ $-\|N_j\| + \|N_j \cap M_i\|$	$\leq$	$\|E_j \cap D_i\|$ $-\|N_j \cap M_i\|$	$\leq$
2.	$i = 1, 2, \ldots, s$ $j = s^{'} + 1, \ldots, m - v + s^{'}$	$D_i - M_i$	E_j	$\frac{\|E_j \cap D_i\|}{\|E_j\|}$	$=$	$\|E_j\| - \|E_j \cap D_i\|$	$=$	$\|E_j \cap D_i\|$	$=$
3.	$i = 1, 2, \ldots, s$ $j = m - v + s^{'} + 1, \ldots, m$	$D_i - M_i$	[b]	[b]	[b]	[b]	[b]	[b]	[b]
4.	$i = s + 1, \ldots, n - u + s$ $j = 1, 2, \ldots, s^{'}$	D_i	$E_j - N_j$	$\frac{\|E_j \cap D_i\|}{\|E_j\| - \|N_j\|}$	$>$	$\|E_j\| - \|N_j\| -$ $\|E_j \cap D_i\|$	$<$	$\|E_j \cap D_i\|$	$=$
5.	$i = s + 1, \ldots, n - u + s$ $j = s^{'} + 1, \ldots, m - v + s^{'}$	D_i	E_j	$\frac{\|E_j \cap D_i\|}{\|E_j\|}$	$=$	$\|E_j\| - \|E_j \cap D_i\|$	$=$	$\|E_j \cap D_i\|$	$=$
6.	$i = s + 1, \ldots, n - u + s$ $j = m - v + s^{'} + 1, \ldots, m$	D_i	[b]	[b]	[b]	[b]	[b]	[b]	[b]
7.	$i = n - u + s + 1, \ldots, n$ $j = 1, 2, \ldots, s^{'}$	[b]	$E_j - N_j$	[b]	[b]	[b]	[b]	[b]	[b]
8.	$i = n - u + s + 1, \ldots, n$ $j = s^{'} + 1, \ldots, m - v + s^{'}$	[b]	E_j	[b]	[b]	[b]	[b]	[b]	[b]
9.	$i = n - u + s + 1, \ldots, n$ $j = m - v + s^{'} + 1, \ldots, m$	[b]	[b]	[b]	[b]	[b]	[b]	[b]	[b]

[a]Denotes that the relationship between the conditional probability of time $t + 1$ and that of time t is uncertain.
[b]Denotes that the deleted information along with the deletion of decision classes and equivalence classes at time $t + 1$.

In Proposition 4.2.2, the prerequisite is that the deleted object does not belong to one original decision class. In the DqII-DTRS model, when the original equivalence class to which the deleted object belongs is contained in the upper (lower) approximation of the original decision class, the upper (lower) approximation of the corresponding current decision class can be updated directly. When this original equivalence class is not contained in the lower approximation of the original decision class, we need to update the lower approximation of the corresponding current decision class as long as the current conditional probability is not less than α.

Example 6 (*Continuation of Example 5*)**.** We elaborate on the incremental approximation updating mechanisms of DqII-DTRS with the deletion of an object, where $k = 1$ and $\alpha = \frac{2}{3}$.

From Example 2, we know that $\overline{R}_k^{II}(D_1) = E_1 \cup E_2$, $\underline{R}_{(\alpha,\beta)}^{II}(D_1) = E_1 \cup E_2$, $\overline{R}_k^{II}(D_2) = E_4$, $\underline{R}_{(\alpha,\beta)}^{II}(D_2) = E_3 \cup E_4$. At time $t + 1$, we know that $D_1' = D_1 = \{x_1, x_2, x_6, x_7\}$, $D_2' = D_2 - \{x_8\} = \{x_3, x_4, x_5\}$ and $E_1' = E_1 = \{x_1, x_3, x_6\}$, $E_2' = E_2 = \{x_2, x_7\}$, $E_3' = E_3 = \{x_4\}$, $E_4' = E_4 - \{x_8\} = \{x_5\}$ from Table 2.

By $x_8 \in E_4$ and $x_8 \in D_2$, we calculate the approximations of D_2' according to Proposition 4.2.1. Because $E_4 \subset \overline{R}_k^{II}(D_2)$, $\overline{g}(E_4', D_2') = 1 \not> k$, $E_4 \subset \underline{R}_{(\alpha,\beta)}^{II}(D_2)$, $P(D_2'|E_4') = 1 \geq \alpha$, we get $\overline{R}_k^{II}(D_2') = \overline{R}_k^{II}(D_2) - E_4$ and $\underline{R}_{(\alpha,\beta)}^{II}(D_2') = \underline{R}_{(\alpha,\beta)}^{II}(D_2) - \{x_8\}$ from the conclusions (1) and (3) of Proposition 4.2.1.

By $x_8 \in E_4$ and $x_8 \notin D_1$, we calculate the approximations of D_1' according to Proposition 4.2.2. Because $E_4 \not\subset \overline{R}_k^{II}(D_1)$, $E_4 \not\subset \underline{R}_{(\alpha,\beta)}^{II}(D_1)$, $P(D_1'|E_4') = 0 \not\geq \alpha$, we get $\overline{R}_k^{II}(D_1') = \overline{R}_k^{II}(D_1)$ and $\underline{R}_{(\alpha,\beta)}^{II}(D_1') = \underline{R}_{(\alpha,\beta)}^{II}(D_1)$ from the conclusions (2) and (4) of Proposition 4.2.2.

4.3. The updating mechanisms for the concept approximations of Dq-DTRS with the batch deletion of objects

In this subsection, we study the approximation updating methods of DqI-DTRS and of DqII-DTRS when many objects are deleted simultaneously at time $t + 1$. Next, we propose the incremental approximation updating mechanisms of DqI-DTRS and DqII-DTRS with the batch deletion of objects.

Let ΔU denote the deleted object set at time $t + 1$, and $\Delta U/D = \{M_1, M_2, \ldots, M_s, M_{s+1}, \ldots, M_u\}$ and $\Delta U/C = \{N_1, N_2, \ldots, N_{s'}, N_{s'+1}, \ldots, N_v\}$ denote decision partition and conditional partition of ΔU, respectively. Then the decision partition of U' is $U'/D = \{D_1', D_2', \ldots, D_s', D_{s+1}', \ldots, D_{n-u+s}'\}$, where for $i = 1, 2, \ldots, s$, $D_i' = D_i - M_i$ denote the original changed decision classes; for $i = s + 1, s + 2, \ldots, n - u + s$, $D_i' = D_i$ denote the original unchanged decision classes. Other decision classes $D_i = M_{i-n+u} (i = n - u + s + 1, n - u + s + 2, \ldots, n)$ are completely removed. And the conditional partition of U' is $U'/C = \{E_1', E_2', \ldots, E_{s'}', E_{s'+1}', \ldots, E_{m-v+s'}'\}$, where for $j = 1, 2, \ldots, s'$, $E_j' = E_j - N_j$ denote the original changed equivalence classes; for $j = s' + 1, s' + 2, \ldots, m - v + s'$, $E_j' = E_j$ denote the original unchanged equivalence classes. Other equivalence classes $E_j = N_{j-m+v}, j = m-v+s'+1, m-v+s'+2, \ldots, m$, are completely removed. In Table 6, we give information about decision classes, equivalence classes, conditional probability, external grade and internal grade when many objects are deleted at the same time.

From Table 6, we know that only the approximations of the decision classes (D_i', $i = 1, \ldots, n - u + s$) in DqI-DTRS and DqII-DTRS models need to be calculated at time $t + 1$.

Proposition 4.3.1. *For the original changed decision classes D_i', $i = 1, 2, \ldots, s$, the following conclusions for $\overline{R}_{(\alpha,\beta)}^{I}(D_i')$ and $\underline{R}_k^{I}(D_i')$ hold.*

(1) For $E_j, j = 1, 2, \ldots, s'$,

(a_1) when $E_j \subseteq \overline{R}_{(\alpha,\beta)}^{I}(D_i)$, if $P(D_i'|E_j') > \beta$, then $\overline{R}_{(\alpha,\beta)}^{I}(D_i') = \overline{R}_{(\alpha,\beta)}^{I}(D_i) - N_j$; otherwise, $\overline{R}_{(\alpha,\beta)}^{I}(D_i') = \overline{R}_{(\alpha,\beta)}^{I}(D_i) - E_j$.

(a_2) when $E_j \not\subset \overline{R}_{(\alpha,\beta)}^{I}(D_i)$, if $P(D_i'|E_j') > \beta$, then $\overline{R}_{(\alpha,\beta)}^{I}(D_i') = \overline{R}_{(\alpha,\beta)}^{I}(D_i) \cup E_j'$; otherwise, $\overline{R}_{(\alpha,\beta)}^{I}(D_i') = \overline{R}_{(\alpha,\beta)}^{I}(D_i)$.

(b_1) when $E_j \subseteq \underline{R}_k^{I}(D_i)$, there is $\underline{R}_k^{I}(D_i') = \underline{R}_k^{I}(D_i) - N_j$.

(b_2) when $E_j \not\subset \underline{R}_k^{I}(D_i)$, if $\underline{g}(E_j', D_i') \leq k$, then $\underline{R}_k^{I}(D_i') = \underline{R}_k^{I}(D_i) \cup E_j'$; otherwise, $\underline{R}_k^{I}(D_i') = \underline{R}_k^{I}(D_i)$.

(2) For $E_j, j = s' + 1, s' + 2, \ldots, m - v + s'$, there are $\overline{R}_{(\alpha,\beta)}^{I}(D_i') = \overline{R}_{(\alpha,\beta)}^{I}(D_i)$ and $\underline{R}_k^{I}(D_i') = \underline{R}_k^{I}(D_i)$.

(3) For $E_j, j = m - v + s' + 1, m - v + s' + 2, \ldots, m$, there are $\overline{R}^{I}_{(\alpha,\beta)}(D'_i) = \overline{R}^{I}_{(\alpha,\beta)}(D_i) - E_j = \overline{R}^{I}_{(\alpha,\beta)}(D_i) - N_{j-m+v}$ and $\underline{R}^{I}_{k}(D'_i) = \underline{R}^{I}_{k}(D_i) - E_j = \underline{R}^{I}_{k}(D_i) - N_{j-m+v}$.

Proposition 4.3.2. *For the original unchanged decision classes D'_i, $i = s+1, s+2, \ldots, n-u+s$, the following conclusions for $\overline{R}^{I}_{(\alpha,\beta)}(D'_i)$ and $\underline{R}^{I}_{k}(D'_i)$ hold.*

(1) For $E_j, j = 1, 2, \ldots, s'$,

- (a_1) when $E_j \subseteq \overline{R}^{I}_{(\alpha,\beta)}(D_i)$, there is $\overline{R}^{I}_{(\alpha,\beta)}(D'_i) = \overline{R}^{I}_{(\alpha,\beta)}(D_i) - N_j$.
- (a_2) when $E_j \not\subset \overline{R}^{I}_{(\alpha,\beta)}(D_i)$, if $P(D'_i|E'_j) > \beta$, then $\overline{R}^{I}_{(\alpha,\beta)}(D'_i) = \overline{R}^{I}_{(\alpha,\beta)}(D_i) \cup E'_j$; otherwise, $\overline{R}^{I}_{(\alpha,\beta)}(D'_i) = \overline{R}^{I}_{(\alpha,\beta)}(D_i)$.
- (b_1) when $E_j \subseteq \underline{R}^{I}_{k}(D_i)$, there is $\underline{R}^{I}_{k}(D'_i) = \underline{R}^{I}_{k}(D_i) - N_j$.
- (b_2) when $E_j \not\subset \underline{R}^{I}_{k}(D_i)$, if $\underline{g}(E'_j, D'_i) \le k$, then $\underline{R}^{I}_{k}(D'_i) = \underline{R}^{I}_{k}(D_i) \cup E'_j$; otherwise, $\underline{R}^{I}_{k}(D'_i) = \underline{R}^{I}_{k}(D_i)$.

(2) For $E_j, j = s'+1, s'+2, \ldots, m-v+s'$, there are $\overline{R}^{I}_{(\alpha,\beta)}(D'_i) = \overline{R}^{I}_{(\alpha,\beta)}(D_i)$ and $\underline{R}^{I}_{k}(D'_i) = \underline{R}^{I}_{k}(D_i)$.

(3) For $E_j, j = m - v + s' + 1, m - v + s' + 2, \ldots, m$, there are $\overline{R}^{I}_{(\alpha,\beta)}(D'_i) = \overline{R}^{I}_{(\alpha,\beta)}(D_i) - N_{j-m+v}$ and $\underline{R}^{I}_{k}(D'_i) = \underline{R}^{I}_{k}(D_i) - N_{j-m+v}$.

According to the formulas (2.1)–(2.2) and the probability and external grade information of Table 6, Propositions 4.3.1–4.3.2 are true.

Example 7. A decision information system $S = (U, A = C \cup D, V, f)$ at time t is shown in Table 2. At time $t + 1$, an object set $\Delta U = \{x_5, x_6, x_7, x_8\}$ is deleted. We elaborate on the incremental approximation updating mechanisms of DqI-DTRS with the batch deletion of objects, where $\beta = \frac{1}{3}$ and $k = 1$.

From Example 1, we know that $\overline{R}^{I}_{(\alpha,\beta)}(D_1) = E_1 \cup E_2$, $\underline{R}^{I}_{k}(D_1) = E_1 \cup E_2 \cup E_3$, $\overline{R}^{I}_{(\alpha,\beta)}(D_2) = E_3 \cup E_4$, $\underline{R}^{I}_{k}(D_2) = E_3 \cup E_4$. From Table 2, there are $\Delta U/D = \{M_1 = \{x_6, x_7\}, M_2 = \{x_5, x_8\}\}$ and $\Delta U/C = \{N_1 = \{x_6\}, N_2 = \{x_7\}, N_3 = \{x_5, x_8\}\}$. Then we can get $U'/D = \{D'_1, D'_2\}$, where the original changed decision classes $D'_1 = D_1 - M_1 = \{x_1, x_2\}$ and $D'_2 = D_2 - M_2 = \{x_3, x_4\}$. Similarly, we obtain $U'/C = \{E'_1, E'_2, E'_3\}$, where the original changed equivalence classes $E'_1 = E_1 - N_1 = \{x_1, x_3\}$ and $E'_2 = E_2 - N_2 = \{x_2\}$, the original unchanged equivalence class $E'_3 = E_3 = \{x_4\}$. It should be noted that the original equivalence class $E_4 = \{x_5, x_8\}$ has been completely removed.

We calculate the approximations of the original changed decision classes D'_1 and D'_2 by Proposition 4.3.1.

By $E_1 \subset \overline{R}^{I}_{(\alpha,\beta)}(D_1)$, $P(D'_1|E'_1) = \frac{1}{2} > \beta$, $E_2 \subset \overline{R}^{I}_{(\alpha,\beta)}(D_1)$ and $P(D'_1|E'_2) = 1 > \beta$, the upper approximation of D'_1 is $\overline{R}^{I}_{(\alpha,\beta)}(D'_1) = \overline{R}^{I}_{(\alpha,\beta)}(D_1) - N_1 - N_2 - E_4 = \{x_1, x_2, x_3\}$ from the conclusions (a_1) and (2–3) of Proposition 4.3.1. Similarly, by $E_1 \subset \underline{R}^{I}_{k}(D_1)$, $E_2 \subset \underline{R}^{I}_{k}(D_1)$, the lower approximation of D'_1 is $\underline{R}^{I}_{k}(D'_1) = \underline{R}^{I}_{k}(D_1) - N_1 - N_2 - E_4 = \{x_1, x_2, x_3, x_4\}$ from the conclusions (b_1) and (2–3) of Proposition 4.3.1.

By $E_1 \not\subset \overline{R}^{I}_{(\alpha,\beta)}(D_2)$, $P(D'_2|E'_1) = \frac{1}{2} > \beta$, $E_2 \not\subset \overline{R}^{I}_{(\alpha,\beta)}(D_2)$ and $P(D'_2|E'_2) = 0 \ngtr \beta$, there is $\overline{R}^{I}_{(\alpha,\beta)}(D'_2) = \overline{R}^{I}_{(\alpha,\beta)}(D_2) \cup E'_1 - E_4$ from the conclusions (a_2) and (2–3) of Proposition 4.3.1. Similarly, by $E_1 \not\subset \underline{R}^{I}_{k}(D_2)$, $\underline{g}(E'_1, D'_2) = 1 \le k$, $E_2 \not\subset \underline{R}^{I}_{k}(D_2)$ and $\underline{g}(E'_2, D'_2) = 1 \le k$, there is $\underline{R}^{I}_{k}(D'_2) = \underline{R}^{I}_{k}(D_2) \cup E'_1 \cup E'_2 - E_4$ from the conclusions (b_2) and (2–3) of Proposition 4.3.1.

Proposition 4.3.3. *For the original changed decision classes D'_i, $i = 1, 2, \ldots, s$, the following conclusions for $\overline{R}^{II}_{k}(D'_i)$ and $\underline{R}^{II}_{(\alpha,\beta)}(D'_i)$ hold.*

(1) For $E_j, j = 1, 2, \ldots, s'$,

- (a_1) when $E_j \subseteq \overline{R}^{II}_{k}(D_i)$, if $\overline{g}(E'_j, D'_i) > k$, then $\overline{R}^{II}_{k}(D'_i) = \overline{R}^{II}_{k}(D_i) - N_j$; otherwise, $\overline{R}^{II}_{k}(D'_i) = \overline{R}^{II}_{k}(D_i) - E_j$.
- (a_2) when $E_j \not\subset \overline{R}^{II}_{k}(D_i)$, there is $\overline{R}^{II}_{k}(D'_i) = \overline{R}^{II}_{k}(D_i)$.
- (b_1) when $E_j \subseteq \underline{R}^{II}_{(\alpha,\beta)}(D_i)$, if $P(D'_i|E'_j) \ge \alpha$, then $\underline{R}^{II}_{(\alpha,\beta)}(D'_i) = \underline{R}^{II}_{(\alpha,\beta)}(D_i) - N_j$; otherwise, $\underline{R}^{II}_{(\alpha,\beta)}(D'_i) = \underline{R}^{II}_{(\alpha,\beta)}(D_i) - E_j$.
- (b_2) when $E_j \not\subset \underline{R}^{II}_{(\alpha,\beta)}(D_i)$, if $P(D'_i|E'_j) \ge \alpha$, then $\underline{R}^{II}_{(\alpha,\beta)}(D'_i) = \underline{R}^{II}_{(\alpha,\beta)}(D_i) \cup E'_j$; otherwise, $\underline{R}^{II}_{(\alpha,\beta)}(D'_i) = \underline{R}^{II}_{(\alpha,\beta)}(D_i)$.

(2) For $E_j, j = s' + 1, s' + 2, \ldots, m - v + s'$, there are $\overline{R}^{II}_{k}(D'_i) = \overline{R}^{II}_{k}(D_i)$ and $\underline{R}^{II}_{(\alpha,\beta)}(D'_i) = \underline{R}^{II}_{(\alpha,\beta)}(D_i)$.

(3) For $E_j, j = m - v + s' + 1, m - v + s' + 2, \ldots, m$, there are $\overline{R}^{II}_{k}(D'_i) = \overline{R}^{II}_{k}(D_i) - N_{j-m+v}$ and $\underline{R}^{II}_{(\alpha,\beta)}(D'_i) = \underline{R}^{II}_{(\alpha,\beta)}(D_i) - N_{j-m+v}$.

Proposition 4.3.4. *For the original unchanged decision classes D'_i, $i = s + 1, s + 2, \ldots, n - u + s$, the following conclusions for $\overline{R}^{II}_{k}(D'_i)$ and $\underline{R}^{II}_{(\alpha,\beta)}(D'_i)$ hold.*

(1) For $E_j, j = 1, 2, \ldots, s'$,

- (a) if $E_j \subseteq \overline{R}^{II}_{k}(D_i)$, then $\overline{R}^{II}_{k}(D'_i) = \overline{R}^{II}_{k}(D_i) - N_j$; otherwise, $\overline{R}^{II}_{k}(D'_i) = \overline{R}^{II}_{k}(D_i)$.
- (b_1) when $E_j \subseteq \underline{R}^{II}_{(\alpha,\beta)}(D_i)$, there is $\underline{R}^{II}_{(\alpha,\beta)}(D'_i) = \underline{R}^{II}_{(\alpha,\beta)}(D_i) - N_j$.
- (b_2) when $E_j \not\subset \underline{R}^{II}_{(\alpha,\beta)}(D_i)$, if $P(D'_i|E'_j) \ge \alpha$, then $\underline{R}^{II}_{(\alpha,\beta)}(D'_i) = \underline{R}^{II}_{(\alpha,\beta)}(D_i) \cup E'_j$; otherwise, $\underline{R}^{II}_{(\alpha,\beta)}(D'_i) = \underline{R}^{II}_{(\alpha,\beta)}(D_i)$.

(2) For $E_j, j = s' + 1, s' + 2, \ldots, m - v + s'$, there are $\overline{R}^{II}_{k}(D'_i) = \overline{R}^{II}_{k}(D_i)$ and $\underline{R}^{II}_{(\alpha,\beta)}(D'_i) = \underline{R}^{II}_{(\alpha,\beta)}(D_i)$.

(3) For $E_j, j = m - v + s' + 1, m - v + s' + 2, \ldots, m$, there are $\overline{R}^{II}_{k}(D'_i) = \overline{R}^{II}_{k}(D_i) - N_{j-m+v}$ and $\underline{R}^{II}_{(\alpha,\beta)}(D_i) = \underline{R}^{II}_{(\alpha,\beta)}(D_i) - N_{j-m+v}$.

According to the formulas (2.3)–(2.4) and the probability and internal grade information of Table 6, Propositions 4.3.3–4.3.4 are true.

Example 8 (*Continuation of Example 7*)**.** We elaborate on the incremental approximation updating mechanisms of DqII-DTRS with the batch deletion of objects, where the parameters are $k = 1$ and $\alpha = \frac{2}{3}$. From Example 2, we know that $\overline{R}^{II}_{k}(D_1) = E_1 \cup E_2$, $\underline{R}^{II}_{(\alpha,\beta)}(D_1) = E_1 \cup E_2$, $\overline{R}^{II}_{k}(D_2) = E_4$, $\underline{R}^{II}_{(\alpha,\beta)}(D_2) = E_3 \cup E_4$. We calculate the approximations of D'_1 and D'_2 by Proposition 4.3.3.

By $E_1 \subset \overline{R}^{II}_{k}(D_1)$, $\overline{g}(E'_1, D'_1) = 1 \ngtr k$, $E_2 \subset \overline{R}^{II}_{k}(D_1)$ and $\overline{g}(E'_2, D'_1) = 1 \ngtr k$, the upper approximation of D'_1 is $\overline{R}^{II}_{k}(D'_1) = \overline{R}^{II}_{k}(D_1) - E_1 - E_2 - E_4$ from the conclusions (a_1) and (2–3) of Proposition 4.3.3. Similarly, by $E_1 \subset \underline{R}^{II}_{(\alpha,\beta)}(D_1)$, $P(D'_1|E'_1) = \frac{1}{2} \ngeq \alpha$, $E_2 \subset \underline{R}^{II}_{(\alpha,\beta)}(D_1)$ and $P(D'_1|E'_2) = 1 \ge \alpha$, the lower approximation of D'_1 is $\underline{R}^{II}_{(\alpha,\beta)}(D'_1) = \underline{R}^{II}_{(\alpha,\beta)}(D_1) - E_1 - N_2 - E_4 = \{x_2\}$ from the conclusions (b_1) and (2–3) of Proposition 4.3.3.

By $E_1 \not\subset \overline{R}^{II}_{k}(D_2)$ and $E_2 \not\subset \overline{R}^{II}_{k}(D_2)$, the upper approximation of D'_2 is $\overline{R}^{II}_{k}(D'_2) = \overline{R}^{II}_{k}(D_2) - E_4$ from the conclusions (a_2) and (2–3) of Proposition 4.3.3. Similarly, by $E_1 \not\subset \underline{R}^{II}_{(\alpha,\beta)}(D_2)$, $P(D'_2|E'_1) = \frac{1}{2} \ngeq \alpha$, $E_2 \not\subset \underline{R}^{II}_{(\alpha,\beta)}(D_2)$ and $P(D'_2|E'_2) = 0 \ngeq \alpha$, the lower approximation of D'_2 is $\underline{R}^{II}_{(\alpha,\beta)}(D'_2) = \underline{R}^{II}_{(\alpha,\beta)}(D_2) - E_4$ from the conclusions (b_2) and (2–3) of Proposition 4.3.3.

5. Static and incremental algorithms for updating approximations of Dq-DTRS in dynamic decision systems

In order to more intuitively demonstrate the feasibility and efficiency of the incremental approximation updating mechanisms of Dq-DTRS, we design static algorithms (as a comparative standard) and incremental algorithms to calculate the approximations of DqI-DTRS and DqII-DTRS at time $t+1$, respectively.

5.1. Static algorithms for computing the approximations of Dq-DTRS models

Two traditional approaches for computing the lower and upper approximations of DqI-DTRS and DqII-DTRS in dynamic decision information systems are given in algorithms 1 and 2.

Algorithm 1: A static algorithm of DqI-DTRS (SA-I)

Input : A decision information system $S^{'}=(U^{'},A,V^{'},f^{'})$, all decision classes $D_i^{'}$, $i=1,2,\cdots,n^{'}$ and parameters β, k.
Output: $\overline{R}^{I}_{(\alpha,\beta)}(D_i^{'})$ and $\underline{R}^{I}_{k}(D_i^{'})$, $i=1,2,\cdots,n^{'}$.

1 **begin**
2   **for** *each* $x\in U^{'}$ **do**
3   **compute:** all equivalence classes $E_j^{'}$, $j=1,2,\cdots,m^{'}$ generated by C in $U^{'}$
4   **end**
5   **for** $i=1,2,\cdots,n^{'}$ **do**
6   **Init:** $\overline{R}^{I}_{(\alpha,\beta)}(D_i^{'})\leftarrow\emptyset$, $\underline{R}^{I}_{k}(D_i^{'})\leftarrow\emptyset$
7   **for** $j=1,2,\cdots,m^{'}$ **do**
8   **compute:**$P(D_i^{'}|E_j^{'})=|E_j^{'}\cap D_i^{'}|/|E_j^{'}|$
9   **compute:**$\underline{g}(E_j^{'},D_i^{'})=|E_j^{'}|-|E_j^{'}\cap D_i^{'}|$
10   **if** $P(D_i^{'}|E_j^{'})>\beta$ **then**
11         $\overline{R}^{I}_{(\alpha,\beta)}(D_i^{'})\leftarrow\overline{R}^{I}_{(\alpha,\beta)}(D_i^{'})\cup E_j^{'}$ **//Compute the upper approximation of DqI-DTRS by formula 2.1**
12   **end**
13   **if** $\underline{g}(E_j^{'},D_i^{'})\leq k$ **then**
14         $\underline{R}^{I}_{k}(D_i^{'})\leftarrow\underline{R}^{I}_{k}(D_i^{'})\cup E_j^{'}$ **//Compute the lower approximation of DqI-DTRS by formula 2.2**
15   **end**
16   **end**
17   **end**
    return : $\overline{R}^{I}_{(\alpha,\beta)}(D_i^{'})$, $\underline{R}^{I}_{k}(D_i^{'})$, $i=1,2,\cdots,n^{'}$
18 **end**

Algorithm 1 computes the approximations of DqI-DTRS from the whole decision information system in a batch, which we call it a static algorithm of DqI-DTRS and abbreviate it as SA-I. In algorithm 1, steps 2–4 compute all equivalence classes generated by C in U' and with the time complexity of $\mathcal{O}(|U'|^2\times|C|)$. Steps 5–17 compute the upper and lower approximations of all decision classes by formulas (2.1) and (2.2), with the time complexity of $\mathcal{O}(|U'/D|\times|U'/C|)$, where $|U'/D|$ and $|U'/C|$ refer to the number of decision classes and equivalence classes in universe U', respectively. Therefore, the time complexity of algorithm 1 is $\mathcal{O}(|U'|^2\times|C|+|U'/D|\times|U'/C|)$.

Algorithm 2 computes the approximations of DqII-DTRS, which we call it a static algorithm of DqII-DTRS and abbreviate it as SA-II. In algorithm 2, steps 2–4 compute all equivalence classes generated by C in U', with the time complexity of $\mathcal{O}(|U'|^2\times|C|)$. Steps 5–17 compute the upper and lower approximations of all decision classes by formulas (2.3) and (2.4), with the time complexity of $\mathcal{O}(|U'/D|\times|U'/C|)$. Hence, the time complexity of algorithm 2 is $\mathcal{O}(|U'|^2\times|C|+|U'/D|\times|U'/C|)$.

Algorithm 2: A static algorithm of DqII-DTRS (SA-II)

Input : A decision information system $S^{'}=(U^{'},A,V^{'},f^{'})$, all decision classes $D_i^{'}$, $i=1,2,\cdots,n^{'}$ and parameters α, k.
Output : $\overline{R}^{II}_{k}(D_i^{'})$ and $\underline{R}^{II}_{(\alpha,\beta)}(D_i^{'})$, $i=1,2,\cdots,n^{'}$.

1 **begin**
2   **for** *each* $x\in U^{'}$ **do**
3   **compute:** all equivalence class $E_j^{'}$, $j=1,2,\cdots,m^{'}$ generated by C in $U^{'}$
4   **end**
5   **for** $i=1,2,\cdots,n^{'}$ **do**
6   **Init:** $\overline{R}^{II}_{k}(D_i^{'})\leftarrow\emptyset$, $\underline{R}^{II}_{(\alpha,\beta)}(D_i^{'})\leftarrow\emptyset$
7   **for** $j=1,2,\cdots,m^{'}$ **do**
8   **compute:** $\overline{g}(E_j^{'},D_i^{'})=|E_j^{'}\cap D_i^{'}|$
9   **compute:** $P(D_i^{'}|E_j^{'})=|E_j^{'}\cap D_i^{'}|/|E_j^{'}|$
10   **if** $\overline{g}(E_j^{'},D_i^{'})>k$ **then**
11         $\overline{R}^{II}_{k}(D_i^{'})\leftarrow\overline{R}^{II}_{k}(D_i^{'})\cup E_j^{'}$ **// Compute the upper approximation of DqII-DTRS by formula 2.3**
12   **end**
13   **if** $P(D_i^{'}|E_j^{'})\geq\alpha$ **then**
14         $\underline{R}^{II}_{(\alpha,\beta)}(D_i^{'})\leftarrow\underline{R}^{II}_{(\alpha,\beta)}(D_i^{'})\cup E_j^{'}$ **// Compute the lower approximation of DqII-DTRS by formula 2.4**
15   **end**
16   **end**
17   **end**
    return : $\overline{R}^{II}_{k}(D_i^{'})$, $\underline{R}^{II}_{(\alpha,\beta)}(D_i^{'})$, $i=1,2,\cdots,n^{'}$
18 **end**

5.2. Incremental algorithms for updating approximations of Dq-DTRS with the sequential variation of objects

Based on the prior knowledge of time t, we propose incremental approximation updating algorithms of DqI-DTRS and DqII-DTRS models with the sequential insertion and deletion of objects at time $t+1$, respectively.

5.2.1. Incremental updating algorithms for DqI-DTRS and DqII-DTRS with the sequential insertion of objects

Based on the incremental updating mechanisms in Sections 3.1 and 3.2, the incremental algorithms of two Dq-DTRS models are first designed in dynamic decision systems with the sequential insertion of objects. Detailed are shown in algorithms 3 and 4.

Algorithm 3 incrementally computes the approximations of DqI-DTRS with the sequential insertion of objects, which we call it an incremental sequential insertion algorithm of DqI-DTRS and abbreviate it as ISIA-I. In algorithm 3, step 3 computes all equivalence classes generated by C in $U\cup\{x^+\}$, with the time complexity of $\mathcal{O}(|U/C|)$, where $|U/C|$ refers to the number of equivalence classes in universe U. Steps 5–44 update the upper and lower approximations of decision classes derived from time t by Propositions 3.1.2 and 3.1.3, with the time complexity of $\mathcal{O}(|U/D|)$, where $|U/D|$ refers to the number of decision classes in universe U. Steps 45–58 compute the approximations of the new decision class by Proposition 3.1.4, with the time complexity of $\mathcal{O}(|U'/C|)$, where $|U'/C|$ refers to the number of equivalence

Algorithm 3: An incremental sequential insertion algorithm of DqI-DTRS (ISIA-I)

Input : (1) The decision information system $S = (U, A = C \cup D, V, f)$ of time t, decision classes $\{D_1, D_2, \cdots, D_n\}$, equivalence classes $\{E_1, E_2, \cdots, E_m\}$, the upper and lower approximations of DqI-DTRS: $\overline{R}^I_{(\alpha,\beta)}(D_i)$ and $\underline{R}^I_k(D_i), i = 1, 2, \cdots, n$;
(2) The inserted sequential object set ΔU_0 of time $t+1$, parameters β, k

Output : The updated approximations after the insertion of ΔU_0: $\overline{R}^I_{(\alpha,\beta)}(D_i^{'})$ and $\underline{R}^I_k(D_i^{'}), i = 1, 2, \cdots, n^{'}$

begin
for $x^+ \in \Delta U_0$ **do**
compute: All equivalence classes $E_1^{'}, E_2^{'}, \cdots, E_{m^{'}}^{'}$ of time $t+1$; let $x^+ \in E_h^{'}$ and $E_h = E_h^{'} - x^+$.
flag $\leftarrow$ true
for $i = 1, 2, \cdots, n$ **do**
if $\forall x \in D_i, f(x^+, d) = f(x, d)$ **then**
$D_i^{'} \leftarrow D_i \cup \{x^+\}$
flag $\leftarrow$ false
if $E_h \subseteq \overline{R}^I_{(\alpha,\beta)}(D_i)$ **then**
$\overline{R}^I_{(\alpha,\beta)}(D_i^{'}) \leftarrow \overline{R}^I_{(\alpha,\beta)}(D_i) \cup \{x^+\}$
else
if $P(D_i^{'}|E_h^{'}) > \beta$ **then**
$\overline{R}^I_{(\alpha,\beta)}(D_i^{'}) \leftarrow \overline{R}^I_{(\alpha,\beta)}(D_i) \cup E_h^{'}$ // Update the upper approximation of the original changed decision class by Proposition 3.1.2
else
$\overline{R}^I_{(\alpha,\beta)}(D_i^{'}) \leftarrow \overline{R}^I_{(\alpha,\beta)}(D_i)$
end
end
if $E_h \subseteq \underline{R}^I_k(D_i)$ **then**
$\underline{R}^I_k(D_i^{'}) \leftarrow \underline{R}^I_k(D_i) \cup \{x^+\}$ //Update the lower approximation of the original changed decision class by Proposition 3.1.2
else
$\underline{R}^I_k(D_i^{'}) \leftarrow \underline{R}^I_k(D_i)$
end
else
$D_i^{'} \leftarrow D_i$
if $E_h \subseteq \overline{R}^I_{(\alpha,\beta)}(D_i)$ **then**
if $P(D_i^{'}|E_h^{'}) > \beta$ **then**
$\overline{R}^I_{(\alpha,\beta)}(D_i^{'}) \leftarrow \overline{R}^I_{(\alpha,\beta)}(D_i) \cup \{x^+\}$ //Update the upper approximation of the original unchanged decision class by Proposition 3.1.3
else
$\overline{R}^I_{(\alpha,\beta)}(D_i^{'}) \leftarrow \overline{R}^I_{(\alpha,\beta)}(D_i) - E_h$
end
else
$\overline{R}^I_{(\alpha,\beta)}(D_i^{'}) \leftarrow \overline{R}^I_{(\alpha,\beta)}(D_i)$
end
if $E_h \subseteq \underline{R}^I_k(D_i)$ **then**
if $\underline{g}(E_h^{'}, D_i^{'}) \leq k$ **then**
$\underline{R}^I_k(D_i^{'}) \leftarrow \underline{R}^I_k(D_i) \cup \{x^+\}$ //Update the lower approximation of the original unchanged decision class by Proposition 3.1.3
else
$\underline{R}^I_k(D_i^{'}) \leftarrow \underline{R}^I_k(D_i) - E_h$
end
else
$\underline{R}^I_k(D_i^{'}) \leftarrow \underline{R}^I_k(D_i)$
end
end
end
if *flag* **then**
$D_{n+1}^{'} \leftarrow \{x^+\}$
if $P(D_{n+1}^{'}|E_h^{'}) > \beta$ **then**
$\overline{R}^I_{(\alpha,\beta)}(D_{n+1}^{'}) \leftarrow E_h^{'}$ //Compute the upper approximation of the new decision class by Proposition 3.1.4
else
$\overline{R}^I_{(\alpha,\beta)}(D_{n+1}^{'}) \leftarrow \emptyset$
end
$\underline{R}^I_k(D_{n+1}^{'}) \leftarrow \emptyset$
for $j = 1, 2, \cdots, m^{'}$ **do**
if $\underline{g}(E_j^{'}, D_{n+1}^{'}) \leq k$ **then**
$\underline{R}^I_k(D_{n+1}^{'}) \leftarrow \underline{R}^I_k(D_{n+1}^{'}) \cup E_j^{'}$ //Compute the lower approximation of the new decision class by Proposition 3.1.4
end
end
end
end
return : $\overline{R}^I_{(\alpha,\beta)}(D_i^{'}), \underline{R}^I_k(D_i^{'}), i = 1, 2, \cdots, n^{'}$
end

Algorithm 4: An incremental sequential insertion algorithm of DqII-DTRS (ISIA-*II*)

```
Input  : (1) The decision information system $S=(U,A=C\cup D,V,f)$ of time $t$, decision classes $\{D_1,D_2,\cdots,D_n\}$, equivalence classes $\{E_1,E_2,\cdots,E_m\}$, the
         upper and lower approximations of DqI-DTRS: $\overline{R}^{II}_k(D_i)$ and $\underline{R}^{II}_{(\alpha,\beta)}(D_i)$, $i=1,2,\cdots,n$;
         (2) The inserted sequential object set $\Delta U_0$ of time $t+1$, parameters $k,\alpha$
Output : The updated approximations after the insertion of $\Delta U_0$: $\overline{R}^{II}_k(D_i^{'})$ and $\underline{R}^{II}_{(\alpha,\beta)}(D_i^{'})$, $i=1,2,\cdots,n^{'}$
begin
    for $x^+\in\Delta U_0$ do
        compute: All equivalence classes $E_1^{'},E_2^{'},\cdots,E_{m^{'}}^{'}$ of time $t+1$; let $x^+\in E_h^{'}$ and $E_h=E_h^{'}-x^+$.
        flag ← true
        for $i=1,2,\cdots,n$ do
            if $\forall x\in D_i, f(x^+,d)=f(x,d)$ then
                $D_i^{'}\leftarrow D_i\cup\{x^+\}$
                flag ← false
                if $E_h\subseteq\overline{R}^{II}_k(D_i)$ then
                    $\overline{R}^{II}_k(D_i^{'})\leftarrow\overline{R}^{II}_k(D_i)\cup\{x^+\}$
                else
                    if $\overline{g}(E_h^{'},D_i^{'})>k$ then
                        $\overline{R}^{II}_k(D_i^{'})\leftarrow\overline{R}^{II}_k(D_i)\cup E_h^{'}$    //Update the upper approximation of the original changed decision class by Proposition 3.2.1
                    else
                        $\overline{R}^{II}_k(D_i^{'})\leftarrow\overline{R}^{II}_k(D_i)$
                    end
                end
                if $E_h\subseteq\underline{R}^{II}_{(\alpha,\beta)}(D_i)$ then
                    $\underline{R}^{II}_{(\alpha,\beta)}(D_i^{'})\leftarrow\underline{R}^{II}_{(\alpha,\beta)}(D_i)\cup\{x^+\}$    //Update the lower approximation of the original changed decision class by Proposition 3.2.1
                else
                    if $P(D_i^{'}|E_h^{'})\geq\alpha$ then
                        $\underline{R}^{II}_{(\alpha,\beta)}(D_i^{'})\leftarrow\underline{R}^{II}_{(\alpha,\beta)}(D_i)\cup E_h^{'}$
                    else
                        $\underline{R}^{II}_{(\alpha,\beta)}(D_i^{'})\leftarrow\underline{R}^{II}_{(\alpha,\beta)}(D_i)$
                    end
                end
            else
                $D_i^{'}\leftarrow D_i$
                if $E_h\subseteq\overline{R}^{II}_k(D_i)$ then
                    $\overline{R}^{II}_k(D_i^{'})=\overline{R}^{II}_k(D_i)\cup\{x^+\}$
                else
                    $\overline{R}^{II}_k(D_i^{'})=\overline{R}^{II}_k(D_i)$    //Update the upper approximation of the original unchanged decision class by Proposition 3.2.2
                end
                if $E_h\subseteq\underline{R}^{II}_{(\alpha,\beta)}(D_i)$ then
                    if $P(D_i^{'}|E_h^{'})\geq\alpha$ then
                        $\underline{R}^{II}_{(\alpha,\beta)}(D_i^{'})\leftarrow\underline{R}^{II}_{(\alpha,\beta)}(D_i)\cup\{x^+\}$
                    else
                        $\underline{R}^{II}_{(\alpha,\beta)}(D_i^{'})\leftarrow\underline{R}^{II}_{(\alpha,\beta)}(D_i)-E_h$    //Update the lower approximation of the original unchanged decision class by Proposition 3.2.2
                    end
                else
                    $\underline{R}^{II}_{(\alpha,\beta)}(D_i^{'})\leftarrow\underline{R}^{II}_{(\alpha,\beta)}(D_i)$
                end
            end
        end
        if flag then
            $D_{n+1}^{'}\leftarrow\{x^+\}$
            if $\overline{g}(E_h^{'},D_{n+1}^{'})>k$ then
                $\overline{R}^{II}_k(D_{n+1}^{'})\leftarrow E_h^{'}$    //Compute the upper approximation of the new decision class by Proposition 3.2.3
            else
                $\overline{R}^{II}_k(D_{n+1}^{'})\leftarrow\emptyset$
            end
            if $P(D_{n+1}^{'}|E_h^{'})\geq\alpha$ then
                $\underline{R}^{II}_{(\alpha,\beta)}(D_{n+1}^{'})\leftarrow E_h^{'}$    //Compute the lower approximation of the new decision class by Proposition 3.2.3
            else
                $\underline{R}^{II}_{(\alpha,\beta)}(D_{n+1}^{'})\leftarrow\emptyset$
            end
        end
    end
    return : $\overline{R}^{II}_k(D_i^{'})$ and $\underline{R}^{II}_{(\alpha,\beta)}(D_i^{'})$, $i=1,2,\cdots,n^{'}$
end
```

Algorithm 5: An incremental sequential deletion algorithm of DqI-DTRS (ISDA-I)

Input : (1) The decision information system $S=(U, A=C\cup D, V, f)$ of time t, decision classes $\{D_1, D_2, \cdots, D_n\}$, equivalence classes $\{E_1, E_2, \cdots, E_m\}$, the upper and lower approximations of DqI-DTRS: $\overline{R}^{I}_{(\alpha,\beta)}(D_i)$ and $\underline{R}^{I}_{k}(D_i)$, $i=1,2,\cdots,n$;
(2) The deleted sequential object set ΔU_0 of time $t+1$, parameters β, k
Output : The updated approximations after the deletion of ΔU_0: $\overline{R}^{I}_{(\alpha,\beta)}(D_i^{'})$ and $\underline{R}^{I}_{k}(D_i^{'})$, $i=1,2,\cdots,n^{'}$

begin
  for $x^- \in \Delta U_0$ **do**
    compute: All equivalence classes $E_1^{'}, E_2^{'}, \cdots, E_{m^{'}}^{'}$ at time $t+1$; let $x^- \in E_h$ and $E_h^{'} = E_h - x^-$.
    for $i = 1, 2, \cdots, n$ **do**
      if $\forall x \in D_i, f(x^-, d) = f(x, d)$ **then**
        $D_i^{'} \leftarrow D_i - \{x^-\}$
        if $E_h \subseteq \overline{R}^{I}_{(\alpha,\beta)}(D_i)$ **then**
          if $P(D_i^{'}|E_h^{'}) > \beta$ **then**
            $\overline{R}^{I}_{(\alpha,\beta)}(D_i^{'}) \leftarrow \overline{R}^{I}_{(\alpha,\beta)}(D_i) - \{x^-\}$ //Update the upper approximation of the original changed decision class by Proposition 4.1.1
          else
            $\overline{R}^{I}_{(\alpha,\beta)}(D_i^{'}) \leftarrow \overline{R}^{I}_{(\alpha,\beta)}(D_i) - E_h$
          end
        else
          $\overline{R}^{I}_{(\alpha,\beta)}(D_i^{'}) \leftarrow \overline{R}^{I}_{(\alpha,\beta)}(D_i)$
        end
        if $E_h \subseteq \underline{R}^{I}_{k}(D_i)$ **then**
          $\underline{R}^{I}_{k}(D_i^{'}) \leftarrow \underline{R}^{I}_{k}(D_i) - \{x^-\}$ //Update the lower approximation of the original changed decision class by Proposition 4.1.1
        else
          $\underline{R}^{I}_{k}(D_i^{'}) \leftarrow \underline{R}^{I}_{k}(D_i)$
        end
      else
        $D_i^{'} \leftarrow D_i$
        if $E_h \subseteq \overline{R}^{I}_{(\alpha,\beta)}(D_i)$ **then**
          $\overline{R}^{I}_{(\alpha,\beta)}(D_i^{'}) \leftarrow \overline{R}^{I}_{(\alpha,\beta)}(D_i) - \{x^-\}$
        else
          if $P(D_i^{'}|E_h^{'}) > \beta$ **then**
            $\overline{R}^{I}_{(\alpha,\beta)}(D_i^{'}) \leftarrow \overline{R}^{I}_{(\alpha,\beta)}(D_i) \cup E_h^{'}$ //Update the upper approximation of the original unchanged decision class by Proposition 4.1.2
          else
            $\overline{R}^{I}_{(\alpha,\beta)}(D_i^{'}) \leftarrow \overline{R}^{I}_{(\alpha,\beta)}(D_i)$
          end
        end
        if $E_h \subseteq \underline{R}^{I}_{k}(D_i)$ **then**
          $\underline{R}^{I}_{k}(D_i^{'}) \leftarrow \underline{R}^{I}_{k}(D_i) - \{x^-\}$
        else
          if $\underline{g}(E_h^{'}, D_i^{'}) \leq k$ **then**
            $\underline{R}^{I}_{k}(D_i^{'}) \leftarrow \underline{R}^{I}_{k}(D_i) \cup E_h^{'}$ //Update the lower approximation of the original unchanged decision class by Proposition 4.1.2
          else
            $\underline{R}^{I}_{k}(D_i^{'}) \leftarrow \underline{R}^{I}_{k}(D_i)$
          end
        end
      end
    end
  end
  return : $\overline{R}^{I}_{(\alpha,\beta)}(D_i^{'})$, $\underline{R}^{I}_{k}(D_i^{'})$, $i=1,2,\cdots,n^{'}$
end

classes in universe U'. Therefore, the time complexity of algorithm 3 is $\mathcal{O}(|\Delta U_0|\times(|U/C|+|U/D|+|U'/C|))$, where $|\Delta U_0|$ refers to the number of objects in ΔU_0.

Algorithm 4 incrementally computes the approximations of DqII-DTRS with the sequential insertion of objects, which we call it an incremental sequential insertion algorithm of DqII-DTRS and abbreviate it as ISIA-II. In algorithm 4, step 3 computes all equivalence classes generated by C in $U\cup\{x^+\}$, and its time complexity is $\mathcal{O}(|U/C|)$. Steps 5–44 update the upper and lower approximations of decision classes derived from time t by Propositions 3.2.1 and 3.2.2, with the time complexity of $\mathcal{O}(|U/D|)$. Steps 45–57 compute the approximations of the new decision class by Proposition 3.2.3, with the time complexity of $\mathcal{O}(1)$. Therefore, the time complexity of algorithm 4 is $\mathcal{O}(|\Delta U_0| \times (|U/C| + |U/D| + 1))$.

Contrasting algorithm 3 (ISIA-I) and algorithm 4 (ISIA-II), we can find that the complexity of computing the upper approximation of the new decision class in steps 46–51 of DqI-DTRS and DqII-DTRS is the same. But the complexity of computing the lower approximation of the new decision class in DqI-DTRS and DqII-DTRS models is different. In steps 52–56 of ISIA-II, only the equivalence class containing x^+ may belong to the lower approximation of the new decision class of DqII-DTRS. However, in steps 52–57 of ISIA-I, in addition to the equivalence class containing x^+, other equivalence classes may also belong to the lower approximation of the new decision class of DqI-DTRS. Therefore, the computation time of DqI-DTRS is more complex than that of DqII-DTRS because of the cyclic judgment of all the current equivalence classes.

5.2.2. Incremental updating algorithms for DqI-DTRS and DqII-DTRS with the sequential deletion of objects

Incremental algorithms 5 and 6 for updating approximations of DqI-DTRS and DqII-DTRS are designed in decision systems with the sequential deletion of objects at time $t+1$, respectively.

Algorithm 5 incrementally updates two approximations of DqI-DTRS with the sequential deletion of objects, which we call it an incremental sequential deletion algorithm of DqI-DTRS and abbreviate it as ISDA-*I*. In algorithm 5, step 3 computes all equivalence classes generated by C in $U-\{x^-\}$, with the time complexity of $\mathcal{O}(|U/C|)$. Steps 4–42 compute two approximations of DqI-DTRS by Propositions 4.1.1 and 4.1.2, and the time complexity is $\mathcal{O}(|U/D|)$. So the time complexity of algorithm 5 is $\mathcal{O}(|\Delta U_0| \times (|U/C| + |U/D|))$.

Algorithm 6 incrementally updates two approximations of DqII-DTRS with the sequential deletion of objects, which we call it an incremental sequential deletion algorithm of DqII-DTRS and abbreviate it as ISDA-*II*. In algorithm 6, step 3 computes all equivalence classes generated by C in $U-\{x^-\}$, and its time complexity is $\mathcal{O}(|U/C|)$. Steps 4–42 compute two approximations of DqII-DTRS by Propositions 4.2.1 and 4.2.2, with the time complexity of $\mathcal{O}(|U/D|)$. So the time complexity of algorithm 6 is $\mathcal{O}(|\Delta U_0| \times (|U/C| + |U/D|))$.

Remark 1. In practical applications, sometimes objects are deleted from and inserted into data simultaneously in a certain order. Under such circumstances, we choose the algorithms (ISIA-*I*, ISIA-*II*, ISDA-*I* and ISDA-*II*) that meets the actual needs to update the approximations of Dq-DTRS according to the order of object change.

5.3. Incremental algorithms for updating approximations of Dq-DTRS with the batch variation of objects

Based on the incremental updating mechanisms for the case of the batch insertion of objects in Section 3.3, the incremental batch insertion algorithms of DqI-DTRS and DqII-DTRS models are designed in dynamic decision systems. Detailed are shown in algorithms 7 and 8.

Algorithm 7 incrementally updates the approximations of DqI-DTRS with the batch insertion of objects, which we call it an incremental batch insertion algorithm of DqI-DTRS and abbreviate it as IBIA-*I*. In algorithm 7, step 2 computes the decision partition and conditional partition on the newly inserted object set ΔU, with the time complexity of $\mathcal{O}(|\Delta U|^2(|D| + |C|))$. Steps 3–6 update the decision partition and conditional partition on U', with the time complexity of $\mathcal{O}(|U/D| \times |\Delta U/D| \times |D| + |U/C| \times |\Delta U/C| \times |C|)$, where $|\Delta U/D|$ and $|\Delta U/C|$ refer to the number of decision classes and equivalence classes in ΔU, respectively. Steps 7–35 update the approximations of the original changed decision classes by Proposition 3.3.1, with the time complexity of $\mathcal{O}(s \times |\Delta U/C|)$, where s refers to the number of the original changed decision classes in universe U. Steps 36–52 update the approximations of the original unchanged decision classes by Proposition 3.3.2, with the time complexity of $\mathcal{O}((n-s) \times |\Delta U/C|)$, where $(n-s)$ refers to the number of the original unchanged decision classes in universe U. Steps 53–55 compute the approximations of the newly inserted decision classes by formulas (2.1)–(2.2), with the time complexity of $\mathcal{O}((u-s) \times |U'/C|)$, where $(u-s)$ refers to the number of the newly inserted decision classes. Therefore, the time complexity of algorithm 7 is $\mathcal{O}(|\Delta U|^2(|D| + |C|) + |U/D| \times |\Delta U/D| \times |D| + |U/C| \times |\Delta U/C| \times |C| + |U/D| \times |\Delta U/C| + (u-s) \times |U'/C|)$.

Algorithm 8 incrementally updates the approximations of DqII-DTRS with the batch insertion of objects, which we call it an incremental batch insertion algorithm of DqII-DTRS and abbreviate it as IBIA-*II*. In algorithm 8, step 2 computes the decision partition and conditional partition on ΔU, with the time complexity of $\mathcal{O}(|\Delta U|^2(|D| + |C|))$. Steps 3–6 update the decision partition and conditional partition on U', with the time complexity of $\mathcal{O}(|U/D| \times |\Delta U/D| \times |D| + |U/C| \times |\Delta U/C| \times |C|)$. Steps 7–35 update the approximations of the original changed decision classes by Proposition 3.3.3, with the time complexity of $\mathcal{O}(s \times |\Delta U/C|)$. Steps 36–53 update the approximations of the original unchanged decision classes by Proposition 3.3.4, with the time complexity of $\mathcal{O}((n-s) \times s')$, where s' refers to the number of the original changed equivalence classes in universe U. Steps 54–56 compute the approximations of the newly inserted decision classes by formulas (2.3)–(2.4), with the time complexity of $\mathcal{O}((u-s) \times |U'/C|)$. Therefore, the time complexity of algorithm 8 is $\mathcal{O}(|\Delta U|^2(|D|+|C|)+|U/D| \times |\Delta U/D| \times |D| + |U/C| \times |\Delta U/C| \times |C| + s \times |\Delta U/C| + (n-s) \times s' + (u-s) \times |U'/C|)$.

Based on the incremental updating mechanisms for the case of the batch insertion of objects in Section 4.3, the incremental batch deletion algorithms of DqI-DTRS and DqII-DTRS are developed in the dynamic decision systems, respectively. Detailed are shown in algorithms 9 and 10.

Algorithm 9 incrementally updates the approximations of DqI-DTRS with the batch deletion of objects, which we call it an incremental batch deletion algorithm of DqI-DTRS and abbreviate it as IBDA-*I*. In algorithm 9, step 2 computes the decision partition and conditional partition on the deleted object set ΔU, with the time complexity of $\mathcal{O}(|\Delta U|^2(|D| + |C|))$. Steps 3–4 update the decision partition and conditional partition on U', with the time complexity of $\mathcal{O}(|U/D| \times |\Delta U/D| \times |D| + |U/C| \times |\Delta U/C| \times |C|)$. Step 5 removes the original completely deleted equivalence classes from the upper and lower approximations of the original decision classes corresponding to the current decision classes in U', with the time complexity of $\mathcal{O}(v - s')$, where $v - s'$ refers to the number of the original completely deleted equivalence classes. Steps 6–31 update the approximations of the original changed decision classes by Proposition 4.3.1, with the time complexity of $\mathcal{O}(s \times s')$, where s and s' refer to the number of the original changed decision classes and the original changed equivalence classes, respectively. Steps 32–45 update the approximations of the original unchanged decision classes by Proposition 4.3.2, with the time complexity of $\mathcal{O}((n-u) \times s')$, where $n-u$ and s' refer to the number of the original unchanged decision classes and the original changed equivalence classes, respectively. So the time complexity of algorithm 9 is $\mathcal{O}(|\Delta U|^2(|D|+|C|)+|U/D| \times |\Delta U/D| \times |D| + |U/C| \times |\Delta U/C| \times |C| + v + (n-u+s-1) \times s')$.

Algorithm 10 incrementally updates the approximations of DqII-DTRS with the batch deletion of objects, which we call it an incremental batch deletion algorithm of DqII-DTRS and abbreviate it as IBDA-*II*. In algorithm 10, step 2 computes the decision partition and conditional partition on ΔU, with the time complexity of $\mathcal{O}(|\Delta U|^2(|D| + |C|))$. Steps 3–4 update the original decision partition and conditional partition on U', with the time complexity of $\mathcal{O}(|U/D| \times |\Delta U/D| \times |D| + |U/C| \times |\Delta U/C| \times |C|)$. Step 5 removes the original completely deleted equivalence classes from the approximations of the original decision classes corresponding to the current decision classes in U', with the time complexity of $\mathcal{O}(v - s')$. Steps 6–31 update the approximations of the original changed decision classes by Proposition 4.3.3, with the time complexity of $\mathcal{O}(s \times s')$. Steps 32–49 update the approximations of the original unchanged decision classes by Proposition 4.3.4, with the time complexity of $\mathcal{O}((n-u) \times s')$. So the time complexity of algorithm 10 is $\mathcal{O}(|\Delta U|^2(|D| + |C|) + |U/D| \times |\Delta U/D| \times |D| + |U/C| \times |\Delta U/C| \times |C| + v + (n-u+s-1) \times s')$.

Remark 2. In practical applications, sometimes objects are deleted from and inserted into data simultaneously. Under such

Algorithm 6: An incremental sequential deletion algorithm of DqII-DTRS (ISDA-*II*)

Input : (1) The decision information system $S=(U,A=C\cup D,V,f)$ of time t, decision classes $\{D_1,D_2,\cdots,D_n\}$, equivalence classes $\{E_1,E_2,\cdots,E_m\}$, the upper and lower approximations of DqII-DTRS: $\overline{R}^{II}_k(D_i)$ and $\underline{R}^{II}_{(\alpha,\beta)}(D_i)$, $i=1,2,\cdots,n$;
(2) The deleted sequential object set ΔU_0 of time $t+1$, parameters k,α

Output : The updated approximations after the deletion of ΔU_0: $\overline{R}^{II}_k(D_i^{'})$ and $\underline{R}^{II}_{(\alpha,\beta)}(D_i^{'})$, $i=1,2,\cdots,n^{'}$

begin
  for $x^-\in\Delta U_0$ **do**
    compute: All equivalence classes $E_1^{'},E_2^{'},\cdots,E^{'}_{m^{'}}$ of time $t+1$; let $x^-\in E_h$ and $E_h^{'}=E_h-x^-$.
    for $i=1,2,\cdots,n$ **do**
      if $\forall x\in D_i, f(x^-,d)=f(x,d)$ **then**
        $D_i^{'}\leftarrow D_i-\{x^-\}$
        if $E_h\subseteq\overline{R}^{II}_k(D_i)$ **then**
          if $\overline{g}(E_h^{'},D_i^{'})>k$ **then**
            $\overline{R}^{I}_{(\alpha,\beta)}(D_i^{'})\leftarrow\overline{R}^{I}_{(\alpha,\beta)}(D_i)-\{x^-\}$ //Update the upper approximation of the original changed decision class by Proposition 4.2.1
          else
            $\overline{R}^{II}_k(D_i^{'})\leftarrow\overline{R}^{II}_k(D_i)-E_h$
          end
        else
          $\overline{R}^{II}_k(D_i^{'})\leftarrow\overline{R}^{II}_k(D_i)$
        end
        if $E_h\subseteq\underline{R}^{II}_{(\alpha,\beta)}(D_i)$ **then**
          if $P(D_i^{'}|E_h^{'})\geq\alpha$ **then**
            $\underline{R}^{II}_{(\alpha,\beta)}(D_i^{'})\leftarrow\underline{R}^{II}_{(\alpha,\beta)}(D_i)-\{x^-\}$ //Update the lower approximation of the original changed decision class by Proposition 4.2.1
          else
            $\underline{R}^{II}_{(\alpha,\beta)}(D_i^{'})\leftarrow\underline{R}^{II}_{(\alpha,\beta)}(D_i)-E_h$
          end
        else
          $\underline{R}^{II}_{(\alpha,\beta)}(D_i^{'})\leftarrow\underline{R}^{II}_{(\alpha,\beta)}(D_i)$
        end
      else
        $D_i^{'}\leftarrow D_i$
        if $E_h\subseteq\overline{R}^{II}_k(D_i)$ **then**
          $\overline{R}^{II}_k(D_i^{'})\leftarrow\overline{R}^{II}_k(D_i)-\{x^-\}$ //Update the upper approximation of the original unchanged decision class by Proposition 4.2.2
        else
          $\overline{R}^{II}_k(D_i^{'})\leftarrow\overline{R}^{II}_k(D_i)$
        end
        if $E_h\subseteq\underline{R}^{II}_{(\alpha,\beta)}(D_i)$ **then**
          $\underline{R}^{II}_{(\alpha,\beta)}(D_i^{'})\leftarrow\underline{R}^{II}_{(\alpha,\beta)}(D_i)-\{x^-\}$
        else
          if $P(D_i^{'}|E_h^{'})\geq\alpha$ **then**
            $\underline{R}^{II}_{(\alpha,\beta)}(D_i^{'})\leftarrow\underline{R}^{II}_{(\alpha,\beta)}(D_i)\cup E_h^{'}$ //Update the lower approximation of the original unchanged decision class by Proposition 4.2.2
          else
            $\underline{R}^{II}_{(\alpha,\beta)}(D_i^{'})\leftarrow\underline{R}^{II}_{(\alpha,\beta)}(D_i)$
          end
        end
      end
    end
  end
  return : $\overline{R}^{II}_k(D_i^{'})$, $\underline{R}^{II}_{(\alpha,\beta)}(D_i^{'})$, $i=1,2,\cdots,n^{'}$
end

circumstances, considering the influence of data storage on computational performance, we can update the approximations of Dq-DTRS with incremental batch deletion algorithms (IBDA-*I* and IBDA-*II*), then get the final approximations with incremental batch insertion algorithms (IBIA-*I* and IBIA-*II*).

6. Experimental evaluation

In this section, we evaluate the performances of our proposed incremental algorithms for updating the approximations of Dq-DTRS with the sequential and batch variations of objects on several category data sets derived from UCI [48]. Static algorithms (SA-*I* and SA-*II*) are used as benchmarks to verify the computational feasibility and effectiveness of the proposed incremental algorithms of DqI-DTRS and DqII-DTRS models. In dynamic decision systems with the sequential and batch variations (insertion and deletion) of objects, the computational efficiency of incremental sequential insertion, batch insertion, sequential deletion and batch deletion algorithms of DqI-DTRS and DqII-DTRS is verified by comparing them with the corresponding static algorithms SA-*I* and SA-*II*, respectively.

6.1. Experimental design

Experiments are performed on a personal computer with 2.6 GHz CPU, 12.0 GB of memory and 64-bit Windows 10, and have been implemented through Java 8. Six data sets are downloaded from UCI, detailed information is shown in Table 7.

Algorithm 7: An incremental batch insertion algorithm of DqI-DTRS (IBIA-I)

Input : (1) The decision information system $S=(U, A=C\cup D, V, f)$ of time t, decision classes $\{D_1, D_2, \cdots, D_n\}$, equivalence classes $\{E_1, E_2, \cdots, E_m\}$, the upper and lower approximations of DqI-DTRS: $\overline{R}^{I}_{(\alpha,\beta)}(D_i)$ and $\underline{R}^{I}_{k}(D_i)$, $i=1,2,\cdots,n$;
(2) The newly inserted object set ΔU of time $t+1$, parameters β, k

Output : The updated approximations after the insertion of ΔU: $\overline{R}^{I}_{(\alpha,\beta)}(D_i^{'})$ and $\underline{R}^{I}_{k}(D_i^{'})$, $i=1,2,\cdots,n^{'}$

1 **begin**
2 **Compute:** $\Delta U/D=\{M_1, M_2, \cdots, M_s, M_{s+1}, \cdots, M_u\}$ and $\Delta U/C=\{N_1, N_2, \cdots, N_{s^{'}}, N_{s^{'}+1}, \cdots, N_v\}$.
3 **Update:** $U^{'}/D=\{D_1^{'}, D_2^{'}, \cdots, D_s^{'}, D_{s+1}^{'}, \cdots, D_n^{'}, D_{n+1}^{'}, \cdots, D_{n+u-s}^{'}\}$, where $D_i^{'}=D_i\cup M_i (i=1,2,\cdots,s)$, $D_i^{'}=D_i (i=s+1, s+2, \cdots, n)$,
4 $D_i^{'}=M_{i-n+s} (i=n+1, n+2, \cdots, n+u-s)$;
5 $U^{'}/C=\{E_1^{'}, E_2^{'}, \cdots, E_{s^{'}}^{'}, E_{s^{'}+1}^{'}, \cdots, E_m^{'}, E_{m+1}^{'}, \cdots, E_{m+v-s^{'}}^{'}\}$, where $E_j^{'}=E_j\cup N_j (j=1,2,\cdots,s^{'})$, $E_j^{'}=E_j (j=s^{'}+1, s^{'}+2, \cdots, m)$,
6 $E_j^{'}=N_{j-m+s^{'}} (j=m+1, m+2, \cdots, m+v-s^{'})$.
7 **for** $i=1,2,\cdots,s$ **do**
8 **for** $j=1,2,\cdots,s^{'}$ **do**
9 **if** $E_j\subseteq \overline{R}^{I}_{(\alpha,\beta)}(D_i)$ **then**
10 **if** $P(D_i^{'}|E_j^{'})>\beta$ **then**
11 $\overline{R}^{I}_{(\alpha,\beta)}(D_i^{'})\leftarrow \overline{R}^{I}_{(\alpha,\beta)}(D_i)\cup N_j$ //Update the upper approximation of the original changed decision class by Proposition 3.3.1
12 **else**
13 $\overline{R}^{I}_{(\alpha,\beta)}(D_i^{'})\leftarrow \overline{R}^{I}_{(\alpha,\beta)}(D_i)-E_j$
14 **end**
15 **else**
16 **if** $P(D_i^{'}|E_j^{'})>\beta$ **then**
17 $\overline{R}^{I}_{(\alpha,\beta)}(D_i^{'})\leftarrow \overline{R}^{I}_{(\alpha,\beta)}(D_i)\cup E_j^{'}$
18 **else**
19 $\overline{R}^{I}_{(\alpha,\beta)}(D_i^{'})\leftarrow \overline{R}^{I}_{(\alpha,\beta)}(D_i)$
20 **end**
21 **end**
22 **if** $E_j\subseteq \underline{R}^{I}_{k}(D_i)$ **then**
23 **if** $\underline{g}(E_j^{'}, D_i^{'})\leq k$ **then**
24 $\underline{R}^{I}_{k}(D_i^{'})\leftarrow \underline{R}^{I}_{k}(D_i)\cup N_j$ //Update the lower approximation of the original changed decision class by Proposition 3.3.1
25 **else**
26 $\underline{R}^{I}_{k}(D_i^{'})\leftarrow \underline{R}^{I}_{k}(D_i)-E_j$
27 **end**
28 **else**
29 $\underline{R}^{I}_{k}(D_i^{'})\leftarrow \underline{R}^{I}_{k}(D_i)$
30 **end**
31 **end**
32 **for** $j=m+1, m+2, \cdots, m+v-s^{'}$ **do**
33 Update $\overline{R}^{I}_{(\alpha,\beta)}(D_i^{'})$ and $\underline{R}^{I}_{k}(D_i^{'})$ according to the conclusion (3) of Proposition 3.3.1
34 **end**
35 **end**
36 **for** $i=s+1, s+2, \cdots, n$ **do**
37 **for** $j=1,2,\cdots,s^{'}$ **do**
38 **if** $E_j\subseteq \overline{R}^{I}_{(\alpha,\beta)}(D_i)$ **then**
39 **if** $P(D_i^{'}|E_j^{'})>\beta$ **then**
40 $\overline{R}^{I}_{(\alpha,\beta)}(D_i^{'})\leftarrow \overline{R}^{I}_{(\alpha,\beta)}(D_i)\cup N_j$
41 **else**
42 $\overline{R}^{I}_{(\alpha,\beta)}(D_i^{'})\leftarrow \overline{R}^{I}_{(\alpha,\beta)}(D_i)-E_j$ //Update the upper approximation of the original unchanged decision class by Proposition 3.3.2
43 **end**
44 **else**
45 $\overline{R}^{I}_{(\alpha,\beta)}(D_i^{'})\leftarrow \overline{R}^{I}_{(\alpha,\beta)}(D_i)$
46 **end**
47 Update $\underline{R}^{I}_{k}(D_i^{'})$ by steps 22-30. //Update the lower approximation of the original unchanged decision class by Proposition 3.3.2
48 **end**
49 **for** $j=m+1, m+2, \cdots, m+v-s^{'}$ **do**
50 Update $\underline{R}^{I}_{k}(D_i^{'})$ according to the conclusion (3) of Proposition 3.3.2
51 **end**
52 **end**
53 **for** $i=n+1, n+2, \cdots, n+u-s$ **do**
54 Compute $\overline{R}^{I}_{(\alpha,\beta)}(D_i^{'})$ and $\underline{R}^{I}_{k}(D_i^{'})$ by formulas 2.1-2.2
55 **end**
return : $\overline{R}^{I}_{(\alpha,\beta)}(D_i^{'})$, $\underline{R}^{I}_{k}(D_i^{'})$, $i=1,2,\cdots,n^{'}$ $(n^{'}=n+u-s)$
56 **end**

In both the sequential variation of objects and the batch variation of objects, the dynamic processes for the cases of inserting and deleting objects in data sets from t to $t+1$ are presented as follows:

Algorithm 8: An incremental batch insertion algorithm of DqII-DTRS (IBIA-*II*)

Input : (1) The decision information system $S = (U, A = C \cup D, V, f)$ of time t, decision classes $\{D_1, D_2, \cdots, D_n\}$, equivalence classes $\{E_1, E_2, \cdots, E_m\}$, the upper and lower approximations of DqI-DTRS: $\overline{R}^{II}_k(D_i)$ and $\underline{R}^{II}_{(\alpha,\beta)}(D_i)$, $i = 1, 2, \cdots, n$;
(2) The newly inserted object set ΔU of time $t+1$, parameters k, α

Output : The updated approximations after the insertion of ΔU: $\overline{R}^{II}_k(D_i^{'})$ and $\underline{R}^{II}_{(\alpha,\beta)}(D_i^{'})$, $i = 1, 2, \cdots, n^{'}$

begin
  Compute: $\Delta U/D = \{M_1, M_2, \cdots, M_s, M_{s+1}, \cdots, M_u\}$ and $\Delta U/C = \{N_1, N_2, \cdots, N_{s^{'}}, N_{s^{'}+1}, \cdots, N_v\}$.
  Update: $U^{'}/D = \{D_1^{'}, D_2^{'}, \cdots, D_s^{'}, D_{s+1}^{'}, \cdots, D_n^{'}, D_{n+1}^{'}, \cdots, D_{n+u-s}^{'}\}$, where $D_i^{'} = D_i \cup M_i (i = 1, 2, \cdots, s)$, $D_i^{'} = D_i (i = s+1, s+2, \cdots, n)$,
  $D_i^{'} = M_{i-n+s} (i = n+1, n+2, \cdots, n+u-s)$;
  $U^{'}/C = \{E_1^{'}, E_2^{'}, \cdots, E_{s^{'}}^{'}, E_{s^{'}+1}^{'}, \cdots, E_m^{'}, E_{m+1}^{'}, \cdots, E_{m+v-s^{'}}^{'}\}$, where $E_j^{'} = E_j \cup N_j (j = 1, 2, \cdots, s^{'})$, $E_j^{'} = E_j (j = s^{'}+1, s^{'}+2, \cdots, m)$,
  $E_j^{'} = N_{j-m+s^{'}} (j = m+1, m+2, \cdots, m+v-s^{'})$.
  for $i = 1, 2, \cdots, s$ **do**
    for $j = 1, 2, \cdots, s^{'}$ **do**
      if $E_j \subseteq \overline{R}^{II}_k(D_i)$ **then**
        $\overline{R}^{II}_k(D_i^{'}) \leftarrow \overline{R}^{II}_k(D_i) \cup N_j$ // Update the upper approximation of the original changed decision class by Proposition 3.3.3
      else
        if $\overline{g}(E_j^{'}, D_i^{'}) > k$ **then**
          $\overline{R}^{II}_k(D_i^{'}) \leftarrow \overline{R}^{II}_k(D_i) \cup E_j^{'}$
        else
          $\overline{R}^{II}_k(D_i^{'}) \leftarrow \overline{R}^{II}_k(D_i)$
        end
      end
      if $E_j \subseteq \underline{R}^{II}_{(\alpha,\beta)}(D_i)$ **then**
        if $P(D_i^{'}|E_j^{'}) \geq \alpha$ **then**
          $\underline{R}^{II}_{(\alpha,\beta)}(D_i^{'}) \leftarrow \underline{R}^{II}_{(\alpha,\beta)}(D_i) \cup N_j$ // Update the lower approximation of the original changed decision class by Proposition 3.3.3
        else
          $\underline{R}^{II}_{(\alpha,\beta)}(D_i^{'}) \leftarrow \underline{R}^{II}_{(\alpha,\beta)}(D_i) - E_j$
        end
      else
        if $P(D_i^{'}|E_j^{'}) \geq \alpha$ **then**
          $\underline{R}^{II}_{(\alpha,\beta)}(D_i^{'}) \leftarrow \underline{R}^{II}_{(\alpha,\beta)}(D_i) \cup E_j^{'}$
        else
          $\underline{R}^{II}_{(\alpha,\beta)}(D_i^{'}) \leftarrow \underline{R}^{II}_{(\alpha,\beta)}(D_i)$
        end
      end
    end
    for $j = m+1, m+2, \cdots, m+v-s^{'}$ **do**
      Update $\overline{R}^{II}_k(D_i^{'})$ and $\underline{R}^{II}_{(\alpha,\beta)}(D_i^{'})$ according to the conclusion (3) of Proposition 3.3.3
    end
  end
  for $i = s+1, s+2, \cdots, n$ **do**
    for $j = 1, 2, \cdots, s^{'}$ **do**
      if $E_j \subseteq \overline{R}^{II}_k(D_i)$ **then**
        $\overline{R}^{II}_k(D_i^{'}) \leftarrow \overline{R}^{II}_k(D_i) \cup N_j$ // Update the upper approximation of the original unchanged decision class by Proposition 3.3.4
      else
        $\overline{R}^{II}_k(D_i^{'}) \leftarrow \overline{R}^{II}_k(D_i)$
      end
      if $E_j \subseteq \underline{R}^{II}_{(\alpha,\beta)}(D_i)$ **then**
        if $P(D_i^{'}|E_j^{'}) \geq \alpha$ **then**
          $\underline{R}^{II}_{(\alpha,\beta)}(D_i^{'}) \leftarrow \underline{R}^{II}_{(\alpha,\beta)}(D_i) \cup N_j$
        else
          $\underline{R}^{II}_{(\alpha,\beta)}(D_i^{'}) \leftarrow \underline{R}^{II}_{(\alpha,\beta)}(D_i) - E_j$ // Update the lower approximation of the original unchanged decision class by Proposition 3.3.4
        end
      else
        $\underline{R}^{II}_{(\alpha,\beta)}(D_i^{'}) \leftarrow \underline{R}^{II}_{(\alpha,\beta)}(D_i)$
      end
    end
  end
  for $i = n+1, n+2, \cdots, n+u-s$ **do**
    Compute $\overline{R}^{II}_k(D_i^{'})$ and $\underline{R}^{II}_{(\alpha,\beta)}(D_i^{'})$ according to formulas 2.3-2.4
  end
  return : $\overline{R}^{II}_k(D_i^{'})$ and $\underline{R}^{II}_{(\alpha,\beta)}(D_i^{'})$, $i = 1, 2, \cdots, n^{'}$ $(n^{'} = n+u-s)$
end

Algorithm 9: An incremental batch deletion algorithm of DqI-DTRS (IBDA-I)

Input : (1) The decision information system $S=(U,A=C\cup D,V,f)$ of time t, decision classes $\{D_1,D_2,\cdots,D_n\}$, equivalence classes $\{E_1,E_2,\cdots,E_m\}$, the upper and lower approximations of DqI-DTRS: $\overline{R}^I_{(\alpha,\beta)}(D_i)$ and $\underline{R}^I_k(D_i)$, $i=1,2,\cdots,n$;
(2) The deleted object set $\varDelta U$ of time $t+1$, parameters β, k

Output : The updated approximations after the deletion of $\varDelta U$: $\overline{R}^I_{(\alpha,\beta)}(D_i^{'})$ and $\underline{R}^I_k(D_i^{'})$, $i=1,2,\cdots,n^{'}$

1 **begin**
2 **Compute:** $\varDelta U/D=\{M_1,M_2,\cdots,M_s,M_{s+1},\cdots,M_u\}$ and $\varDelta U/C=\{N_1,N_2,\cdots,N_{s^{'}},N_{s^{'}+1},\cdots,N_v\}$.
3 **Update:** $U^{'}/D=\{D_1^{'},D_2^{'},\cdots,D_s^{'},D_{s+1}^{'},\cdots,D_{n-u+s}^{'}\}$, where $D_i^{'}=D_i-M_i\quad(i=1,2,\cdots,s)$, $D_i^{'}=D_i(i=s+1,s+2,\cdots,n-u+s)$;
4 $U^{'}/C=\{E_1^{'},E_2^{'},\cdots,E_{s^{'}}^{'},E_{s^{'}+1}^{'},\cdots,E_{m-v+s^{'}}^{'}\}$, where $E_j^{'}=E_j-N_j(j=1,2,\cdots,s^{'})$, $E_j^{'}=E_j(j=s^{'}+1,s^{'}+2,\cdots,m-v+s^{'})$.
5 Remove N_w, $w=s^{'}+1,s^{'}+2,\ldots,v$ from $\overline{R}^I_{(\alpha,\beta)}(D_i)$ and $\underline{R}^I_k(D_i)$, $i=1,2,\cdots,n-u+s$
6 **for** $i=1,2,\cdots,s$ **do**
7 **for** $j=1,2,\cdots,s^{'}$ **do**
8 **if** $E_j\subseteq\overline{R}^I_{(\alpha,\beta)}(D_i)$ **then**
9 **if** $P(D_i^{'}|E_j^{'})>\beta$ **then**
10 $\overline{R}^I_{(\alpha,\beta)}(D_i^{'})\leftarrow\overline{R}^I_{(\alpha,\beta)}(D_i)-N_j$ //Update the upper approximation of the original changed decision class by Proposition 4.3.1
11 **else**
12 $\overline{R}^I_{(\alpha,\beta)}(D_i^{'})\leftarrow\overline{R}^I_{(\alpha,\beta)}(D_i)-E_j$
13 **end**
14 **else**
15 **if** $P(D_i^{'}|E_j^{'})>\beta$ **then**
16 $\overline{R}^I_{(\alpha,\beta)}(D_i^{'})\leftarrow\overline{R}^I_{(\alpha,\beta)}(D_i)\cup E_j^{'}$
17 **else**
18 $\overline{R}^I_{(\alpha,\beta)}(D_i^{'})\leftarrow\overline{R}^I_{(\alpha,\beta)}(D_i)$
19 **end**
20 **end**
21 **if** $E_j\subseteq\underline{R}^I_k(D_i)$ **then**
22 $\underline{R}^I_k(D_i^{'})\leftarrow\underline{R}^I_k(D_i)-N_j$
23 **else**
24 **if** $\underline{g}(E_j^{'},D_i^{'})\le k$ **then**
25 $\underline{R}^I_k(D_i^{'})\leftarrow\underline{R}^I_k(D_i)\cup E_j^{'}$ //Update the lower approximation of the original changed decision class by Proposition 4.3.1
26 **else**
27 $\underline{R}^I_k(D_i^{'})\leftarrow\underline{R}^I_k(D_i)$
28 **end**
29 **end**
30 **end**
31 **end**
32 **for** $i=s+1,s+2,\cdots,n-u+s$ **do**
33 **for** $j=1,2,\cdots,s^{'}$ **do**
34 **if** $E_j\subseteq\overline{R}^I_{(\alpha,\beta)}(D_i)$ **then**
35 $\overline{R}^I_{(\alpha,\beta)}(D_i^{'})\leftarrow\overline{R}^I_{(\alpha,\beta)}(D_i)-N_j$
36 **else**
37 **if** $P(D_i^{'}|E_j^{'})>\beta$ **then**
38 $\overline{R}^I_{(\alpha,\beta)}(D_i^{'})\leftarrow\overline{R}^I_{(\alpha,\beta)}(D_i)\cup E_j^{'}$ //Update the upper approximation of the original unchanged decision class by Proposition 4.3.2
39 **else**
40 $\overline{R}^I_{(\alpha,\beta)}(D_i^{'})\leftarrow\overline{R}^I_{(\alpha,\beta)}(D_i)$
41 **end**
42 **end**
43 Update $\underline{R}^I_k(D_i^{'})$ by steps 21-29. //Update the lower approximation of the original unchanged decision class by Proposition 4.3.2
44 **end**
45 **end**
return : $\overline{R}^I_{(\alpha,\beta)}(D_i^{'})$, $\underline{R}^I_k(D_i^{'})$, $i=1,2,\cdots,n^{'}$ $(n^{'}=n-u+s)$
46 **end**

(1) In the insertion experiments, we select 50% objects from each data set as original data sets of time t. At time $t+1$, we set the inserting ratio from 10% to 100% in steps of 10% from the remaining objects of each data set.

(2) In the deletion experiments, we select each data set as original data sets of time t. At time $t+1$, we set the deleting ratio from 5% to 50% in steps of 5% from each original data set.

For the parameters of DqI-DTRS and DqII-DTRS, considering the medium error tolerance acceptance level, we first set them to $\beta = 0.3$, $k = 2$ and $\alpha = 0.8$. Meanwhile, we give a detailed analysis of the influence of different performance values on the computational performances of incremental algorithms in Section 6.4. In the following Sections 6.2–6.3, the comparative

Table 7
Data description.

No.	Data sets	Abbreviation	Samples	Attributes	Classes
1	Chess	Che.	3 196	36	2
2	Mushroom	Mus.	8 124	22	2
3	Nursery	Nur.	12 960	8	5
4	Letter recognition	Let.	20 000	16	26
5	Default of credit card clients	Def.	30 000	23	2
6	Bank marketing	Ban.	45 211	16	2

Algorithm 10: An incremental batch deletion algorithm of DqII-DTRS (IBDA-*II*)

Input : (1) The decision information system $S=(U,A=C\cup D,V,f)$ of time t, decision classes $\{D_1,D_2,\cdots,D_n\}$, equivalence classes $\{E_1,E_2,\cdots,E_m\}$, the upper and lower approximations of DqII-DTRS: $\overline{R}_k^{II}(D_i)$ and $\underline{R}_{(\alpha,\beta)}^{II}(D_i)$, $i=1,2,\cdots,n$;
(2) The deleted object set $\varDelta U$ of time $t+1$, parameters k,α

Output : The updated approximations after the deletion of $\varDelta U$: $\overline{R}_k^{II}(D_i^{'})$ and $\underline{R}_{(\alpha,\beta)}^{II}(D_i^{'})$, $i=1,2,\cdots,n^{'}$

begin
  Compute: $\varDelta U/D=\{M_1,M_2,\cdots,M_s,M_{s+1},\cdots,M_u\}$ and $\varDelta U/C=\{N_1,N_2,\cdots,N_{s^{'}},N_{s^{'}+1},\cdots,N_v\}$.
  Update: $U^{'}/D=\{D_1^{'},D_2^{'},\cdots,D_s^{'},D_{s+1}^{'},\cdots,D_{n-u+s}^{'}\}$, where $D_i^{'}=D_i-M_i\ \ (i=1,2,\cdots,s)$, $D_i^{'}=D_i(i=s+1,s+2,\cdots,n-u+s)$;
  $U^{'}/C=\{E_1^{'},E_2^{'},\cdots,E_{s^{'}}^{'},E_{s^{'}+1}^{'},\cdots,E_{m-v+s^{'}}^{'}\}$, where $E_j^{'}=E_j-N_j(j=1,2,\cdots,s^{'})$, $E_j^{'}=E_j(j=s^{'}+1,s^{'}+2,\cdots,m-v+s^{'})$.
  Remove N_w, $w=s^{'}+1,s^{'}+2,\ldots,v$ from $\overline{R}_k^{II}(D_i^{'})$ and $\underline{R}_{(\alpha,\beta)}^{II}(D_i^{'})$, $i=1,2,\cdots,n-u+s$
  for $i=1,2,\cdots,s$ **do**
    for $j=1,2,\cdots,s^{'}$ **do**
      if $E_j\subseteq\overline{R}_k^{II}(D_i)$ **then**
        if $\overline{g}(E_j^{'},D_i^{'})>k$ **then**
          $\overline{R}_k^{II}(D_i^{'})\leftarrow\overline{R}_k^{II}(D_i)-N_j$
        else
          $\overline{R}_k^{II}(D_i^{'})\leftarrow\overline{R}_k^{II}(D_i)-E_j$ //Update the upper approximation of the original changed decision class by Proposition 4.3.3
        end
      else
        $\overline{R}_k^{II}(D_i^{'})\leftarrow\overline{R}_k^{II}(D_i)$
      end
      if $E_j\subseteq\underline{R}_{(\alpha,\beta)}^{II}(D_i)$ **then**
        if $P(D_i^{'}|E_j^{'})\geq\alpha$ **then**
          $\underline{R}_{(\alpha,\beta)}^{II}(D_i^{'})\leftarrow\underline{R}_{(\alpha,\beta)}^{II}(D_i)-N_j$
        else
          $\underline{R}_{(\alpha,\beta)}^{II}(D_i^{'})\leftarrow\underline{R}_{(\alpha,\beta)}^{II}(D_i)-E_j$
        end
      else
        if $P(D_i^{'}|E_j^{'})\geq\alpha$ **then**
          $\underline{R}_{(\alpha,\beta)}^{II}(D_i^{'})\leftarrow\underline{R}_{(\alpha,\beta)}^{II}(D_i)\cup E_j^{'}$ //Update the lower approximation of the original changed decision class by Proposition 4.3.3
        else
          $\underline{R}_{(\alpha,\beta)}^{II}(D_i^{'})\leftarrow\underline{R}_{(\alpha,\beta)}^{II}(D_i)$
        end
      end
    end
  end
  for $i=s+1,s+2,\cdots,n-u+s$ **do**
    for $j=1,2,\cdots,s^{'}$ **do**
      if $E_j\subseteq\overline{R}_k^{II}(D_i)$ **then**
        $\overline{R}_k^{II}(D_i^{'})\leftarrow\overline{R}_k^{II}(D_i)-N_j$ //Update the upper approximation of the original unchanged decision class by Proposition 4.3.4
      else
        $\overline{R}_k^{II}(D_i^{'})\leftarrow\overline{R}_k^{II}(D_i)$
      end
      if $E_j\subseteq\underline{R}_{(\alpha,\beta)}^{II}(D_i)$ **then**
        $\underline{R}_{(\alpha,\beta)}^{II}(D_i^{'})\leftarrow\underline{R}_{(\alpha,\beta)}^{II}(D_i)-N_j$
      else
        if $P(D_i^{'}|E_j^{'})\geq\alpha$ **then**
          $\underline{R}_{(\alpha,\beta)}^{II}(D_i^{'})\leftarrow\underline{R}_{(\alpha,\beta)}^{II}(D_i)\cup E_j^{'}$ //Update the lower approximation of the original unchanged decision class by Proposition 4.3.4
        else
          $\underline{R}_{(\alpha,\beta)}^{II}(D_i^{'})\leftarrow\underline{R}_{(\alpha,\beta)}^{II}(D_i)$
        end
      end
    end
  end
  return : $\overline{R}_k^{II}(D_i^{'})$ and $\underline{R}_{(\alpha,\beta)}^{II}(D_i^{'})$, $i=1,2,\cdots,n^{'}$ $(n^{'}=n-u+s)$
end

experiments between incremental algorithms and static algorithms of DqI-DTRS and DqII-DTRS are first carried out.

6.2. Comparisons of static and incremental algorithms of DqI-DTRS in decision systems with the variation of objects

This subsection evaluates the effectiveness of incremental algorithms of DqI-DTRS with the variation of objects on six data sets from the perspective of computational time of approximations of all decision classes. It must be pointed out that computational time is measured in seconds.

In the case of inserting objects into data, we illustrate the computational efficiency of the incremental sequential and batch insertion algorithms of DqI-DTRS (ISIA-*I* and IBIA-*I*) by comparing them with the static algorithm of DqI-DTRS (SA-*I*). Detailed

Table 8
Comparisons of computational time about the static algorithm and incremental insertion algorithms of DqI-DTRS in inserting objects.

IR (%)	Che.			Mus.			Nur.			Let.			Def.			Ban.		
	SA-*I*	ISIA-*I*	IBIA-*I*	SA-*I*	ISIA-*I*	IBIA-*I*	SA-*I*	ISIA-*I*	IBIA-*I*	SA-*I*	ISIA-*I*	IBIA-*I*	SA-*I*	ISIA-*I*	IBIA-*I*	SA-*I*	ISIA-*I*	IBIA-*I*
10	0.136	0.021	0.028	0.460	0.046	0.056	2.660	0.120	0.143	13.012	0.545	0.507	21.669	0.781	1.038	42.107	1.781	1.695
20	0.296	0.027	0.035	0.980	0.107	0.136	5.620	0.310	0.260	21.200	1.319	1.149	25.966	1.796	1.922	52.216	4.297	3.542
30	0.480	0.039	0.041	1.544	0.186	0.194	8.980	0.562	0.510	29.960	2.254	1.803	31.326	3.328	2.865	64.591	7.234	5.798
40	0.686	0.059	0.058	2.170	0.286	0.242	12.481	0.935	0.801	41.189	3.394	3.099	37.576	5.140	4.213	79.388	12.547	8.273
50	0.936	0.075	0.072	2.892	0.396	0.393	16.574	1.452	1.256	54.084	5.002	4.098	44.966	7.843	5.808	96.873	19.031	11.474
60	1.216	0.091	0.083	3.664	0.560	0.515	20.794	2.124	1.651	68.772	6.939	5.036	53.419	11.093	7.127	116.951	27.703	15.021
70	1.676	0.123	0.111	4.504	0.825	0.763	25.140	2.949	1.929	85.616	9.047	6.021	63.200	15.187	8.960	139.982	40.016	18.391
80	2.076	0.139	0.134	5.394	1.066	0.935	29.670	4.000	2.223	103.293	11.595	7.054	74.325	20.187	10.540	167.638	55.516	21.734
90	2.446	0.151	0.142	6.380	1.346	1.288	34.330	5.223	2.542	124.252	14.809	8.178	86.669	26.390	12.257	198.185	70.531	25.381
100	2.880	0.171	0.167	7.510	1.746	1.462	39.100	6.690	2.888	147.429	18.473	9.326	100.685	32.890	13.986	234.091	88.328	29.519

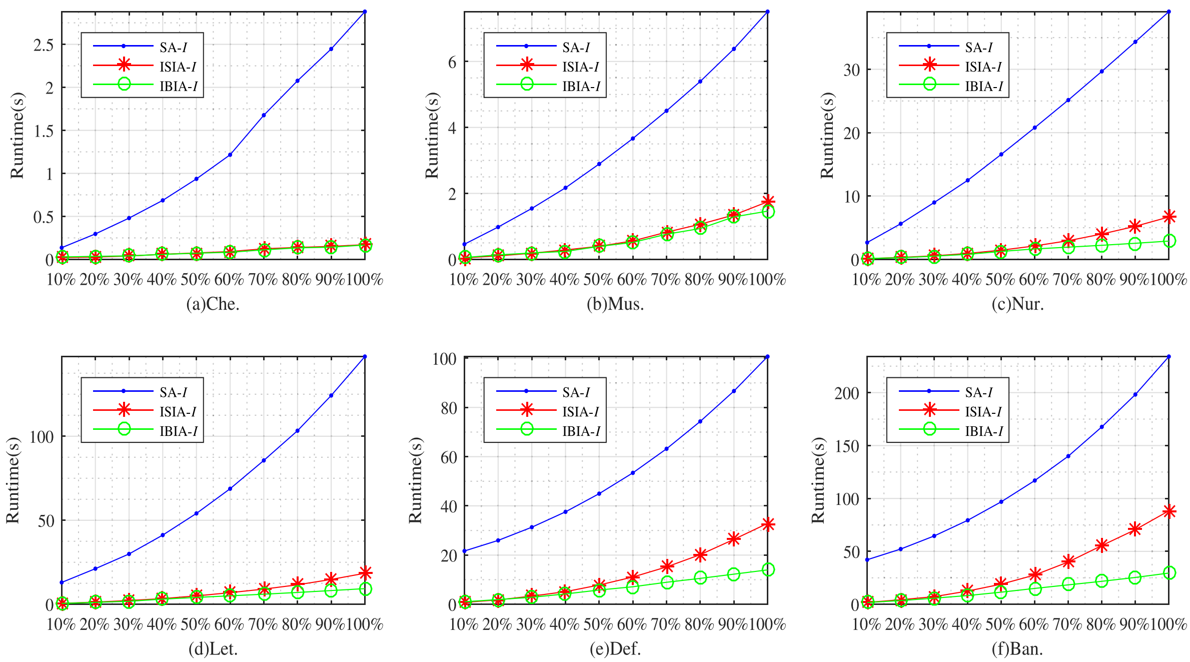

Fig. 1. Comparisons of runtime in static and incremental insertion algorithms of DqI-DTRS versus different inserting ratios.

experimental results are shown in Table 8, where IR denotes the inserting ratio.

Table 8 shows that in comparison with SA-*I*, both ISIA-*I* and IBIA-*I* greatly reduce the runtime of computing approximations from each data set.

First, we compare the performance of ISIA-*I* and SA-*I* to find a better way to update the approximations of DqI-DTRS in the case of the sequential insertion of objects. On data set Che., the runtime of ISIA-*I* shows a decrease up to 15.44%, 9.12%, 8.13%, 8.60%, 8.01%, 7.48%, 7.34%, 6.70%, 6.17%, 5.94% of that of SA-*I* versus different inserting ratios, respectively. On the remaining five data sets Mus., Nur., Let., Def. and Ban., the runtime of ISIA-*I* is reduced to 23.25%, 17.11%, 12.53%, 32.67% and 37.73% at least of the time of SA-*I* at inserting ratio 100%, respectively. On the remaining five data sets, the runtime of ISIA-*I* is reduced by up to 10.00%, 4.51%, 4.19%, 3.60% and 4.23% of the time of SA-*I* at inserting ratio 10%, respectively. The above results imply that ISIA-*I* updates knowledge faster than SA-*I* on data sets with the sequential insertion of many objects.

Then we compare the performance of IBIA-*I*, ISIA-*I* and SA-*I* to find a better way to update the approximations of DqI-DTRS in the case of the batch insertion of objects. At each inserting ratio on each data set, the computational time of IBIA-*I* is less than that of SA-*I*. On the six data sets, the runtime of IBIA-*I* is reduced by up to 5.80%, 11.15%, 4.63%, 3.90%, 4.79% and 4.03% of the time of SA-*I* at the corresponding inserting ratios 100%, 40%, 20%, 10%, 10%, 10%, respectively. The performance of IBIA-*I* is far better than that of SA-*I*. Then we compare the performance of IBIA-*I* and ISIA-*I*. In the ten experiments of the above six data sets, the number of times that IBIA-*I* performs better than ISIA-*I* is 7, 7, 9, 10, 8 and 10, respectively. On Che. and Mus., the performance of IBIA-*I* is slightly better than that of ISIA-*I*. On Nur., IBIA-*I* performs better than ISIA-*I*. On Let., Def. and Ban., the performance of IBIA-*I* is far better than that of ISIA-*I*. So in dynamic data sets with the batch insertion of objects, our optimal choice is IBIA-*I* for calculating the upper and lower approximations of DqI-DTRS.

More intuitive comparisons are shown in Fig. 1. Fig. 1 depicts the detailed changes of the runtime of static and incremental algorithms of DqI-DTRS (SA-*I*, ISIA-*I* and IBIA-*I*) with the increase of inserting ratios. Obviously, at each inserting ratio, the computational speed of ISIA-*I* and IBIA-*I* is faster than that of SA-*I*. Moreover, on each data set with the increase of the inserting ratio, the computational time of SA-*I* increases sharply, that of ISIA-*I* and IBIA-*I* increases more slowly. In addition, IBIA-*I* performs better than ISIA-*I* in many cases.

Therefore, incremental insertion algorithms ISIA-*I* and IBIA-*I* are very efficient in dynamic maintenance of approximations of DqI-DTRS with the sequential and batch insertion of objects.

Table 9
Comparisons of computational time about the static algorithm and incremental deletion algorithms of DqI-DTRS in deleting objects.

DR (%)	Che.			Mus.			Nur.			Let.			Def.			Ban.		
	SA-*I*	ISDA-*I*	IBDA-*I*	SA-*I*	ISDA-*I*	IBDA-*I*	SA-*I*	ISDA-*I*	IBDA-*I*	SA-*I*	ISDA-*I*	IBDA-*I*	SA-*I*	ISDA-*I*	IBDA-*I*	SA-*I*	ISDA-*I*	IBDA-*I*
5	2.470	0.073	0.034	6.829	0.245	0.132	38.829	0.382	0.364	129.368	5.586	0.716	88.840	4.797	1.780	195.247	13.594	3.330
10	2.014	0.114	0.053	5.841	0.466	0.255	34.094	0.810	0.636	108.622	10.640	1.172	76.262	9.844	3.154	165.216	25.734	5.119
15	1.661	0.139	0.071	4.948	0.676	0.367	29.589	1.298	0.981	90.460	15.086	1.580	64.762	14.266	4.317	135.466	38.188	6.739
20	1.360	0.169	0.089	4.062	0.890	0.469	24.989	1.734	1.260	73.768	19.159	1.960	54.887	18.531	5.370	112.481	49.562	8.197
25	1.090	0.200	0.105	3.309	1.086	0.574	20.509	2.229	1.550	59.096	22.765	2.324	46.293	21.984	6.553	92.309	59.187	9.602
30	0.840	0.233	0.126	2.638	1.234	0.687	16.549	2.650	1.765	46.109	26.202	2.644	38.949	25.609	7.447	80.606	67.562	10.847
35	0.636	0.252	0.140	2.007	1.366	0.783	12.753	3.034	1.941	34.893	29.130	2.928	32.652	28.625	8.485	65.747	74.906	11.993
40	0.440	0.269	0.153	1.438	1.489	0.856	9.253	3.510	2.117	25.619	31.763	3.189	27.418	31.453	9.422	53.325	81.484	12.995
45	0.285	0.280	0.163	0.909	1.609	0.912	5.869	3.913	2.292	17.499	34.177	3.431	22.918	33.703	10.085	43.184	87.797	13.869
50	0.150	0.288	0.173	0.420	1.705	0.962	2.649	4.320	2.428	10.182	36.306	3.631	18.574	35.859	10.665	35.044	94.016	14.664

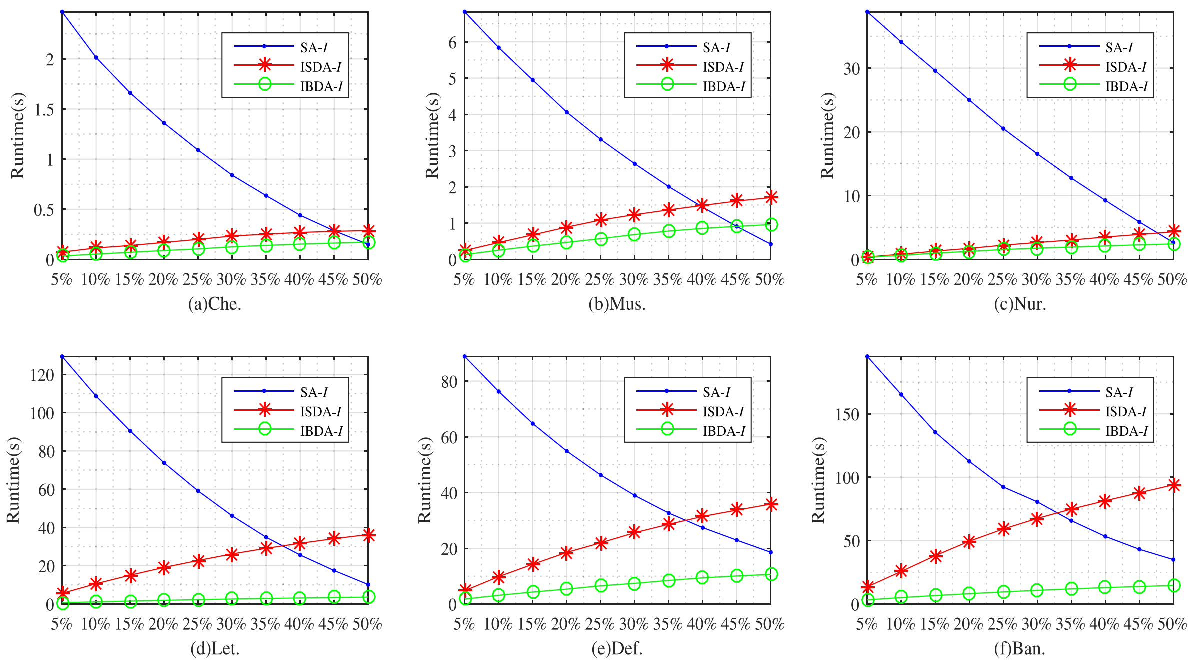

Fig. 2. Comparisons of runtime in static and incremental deletion algorithms of DqI-DTRS versus different deleting ratios.

In the case of deleting objects from data, the computational efficiency of the incremental sequential and batch deletion algorithms of DqI-DTRS (ISDA-*I* and IBDA-*I*) is verified by comparing them with the static algorithm of DqI-DTRS (SA-*I*). Detailed experimental results are shown in Table 9, where DR denotes the deleting ratio.

From Table 9, in comparison with SA-*I*, both ISDA-*I* and IBDA-*I* reduce the runtime of computing approximations to some extent on each data set.

First, we compare ISDA-*I* and SA-*I* to find a better way to update the approximations of DqI-DTRS in the case of the sequential deletion of objects. In the process of increasing the deleting ratio from 5% to 30% in steps of 5%, we find that the runtime of ISDA-*I* is reduced by up to 2.96%, 3.59%, 0.98%, 4.32%, 5.40% and 6.96% of that of SA-*I* at inserting ratio 5%, respectively. Moreover, in the first six experiments, the runtime of ISDA-*I* is always less than that of SA-*I* for each data set. For data sets Che. and Nur., only when the deleting ratio is 50%, the runtime of ISDA-*I* is slightly larger than that of SA-*I*. For data set Mus., when the deleting ratio greater than 40%, the runtime of ISDA-*I* is slightly larger than that of SA-*I*. On data sets Let. and Def., when the deleting ratio is not less than 40%, the runtime of ISDA-*I* is larger than that of SA-*I*. On the last data set Ban., when the deleting ratio is not less than 35%, the runtime of ISDA-*I* is larger than that of SA-*I*. Therefore, ISDA-*I* is very efficient when the amount of deleted objects is small (not more than 35% to 40% of the original data) based on the above analysis.

Then we compare IBDA-*I*, ISDA-*I* and SA-*I* to find a better way to update the approximations of DqI-DTRS in the case of the batch deletion of objects. In the ten experiments of the six data sets, the number of times that IBDA-*I* performs better than SA-*I* is 9, 8, 10, 10, 10 and 10, respectively. On Che., only when the deleting ratio is 50%, the runtime of IBDA-*I* is larger than that of SA-*I*. On Mus., the runtime of IBDA-*I* is slightly larger than that of SA-*I* at the deleting ratio 45% and the runtime of IBDA-*I* is larger than that of SA-*I* at the deleting ratio 50%. On the six data sets, the runtime of IBDA-*I* is reduced by up to 1.38%, 1.93%, 0.94%, 0.55%, 2.00%, 1.71% of the time of SA-*I* at the deleting ratio 5%, respectively. Meanwhile, at each deleting ratio on each data set, the computation time of IBDA-*I* is less than that of ISDA-*I*. On the six data sets, the runtime of IBDA-*I* is reduced by up to 46.49%, 52.70%, 56.20%, 10.00%, 28.98%, 15.60% of the time of ISDA-*I* at the corresponding deleting ratios 10%, 20%, 50%, 50%, 20%, 50%, respectively. So our optimal choice is IBDA-*I* for calculating the approximations of DqI-DTRS in dynamic data sets with the batch deletion of objects.

More intuitive comparisons are shown in Fig. 2. Fig. 2 depicts the detailed changes of the runtime of static and incremental deletion algorithms of DqI-DTRS (SA-*I*, ISDA-*I*, IBDA-*I*) with the increase of deleting ratios. We find that on each data set with

Table 10
Comparisons of computational time about the static algorithm and incremental insertion algorithms of DqII-DTRS in inserting objects.

IR (%)	Che.			Mus.			Nur.			Let.			Def.			Ban.		
	SA-*II*	ISIA-*II*	IBIA-*II*	SA-*II*	ISIA-*II*	IBIA-*II*	SA-*II*	ISIA-*II*	IBIA-*II*	SA-*II*	ISIA-*II*	IBIA-*II*	SA-*II*	ISIA-*II*	IBIA-*II*	SA-*II*	ISIA-*II*	IBIA-*II*
10	0.076	0.020	0.028	0.304	0.020	0.032	0.780	0.043	0.076	12.721	0.627	0.600	19.940	0.672	1.252	41.448	1.891	1.794
20	0.166	0.030	0.035	0.640	0.070	0.077	1.660	0.113	0.169	20.836	1.046	0.940	24.330	1.609	2.145	51.604	4.453	4.165
30	0.266	0.043	0.044	1.060	0.132	0.098	2.650	0.197	0.231	30.418	1.889	1.531	29.674	3.203	3.127	64.026	7.703	6.730
40	0.385	0.059	0.053	1.540	0.204	0.152	3.700	0.323	0.304	41.520	2.975	2.727	35.924	5.219	4.609	79.182	12.500	9.385
50	0.526	0.064	0.061	2.060	0.292	0.268	4.800	0.497	0.441	54.334	4.357	4.052	43.283	8.500	5.853	96.698	19.219	12.021
60	0.685	0.080	0.075	2.660	0.440	0.381	6.060	0.693	0.674	68.849	6.037	5.165	51.846	12.156	7.156	117.182	29.032	15.335
70	0.866	0.090	0.082	3.300	0.646	0.567	7.470	0.913	0.853	85.289	7.840	6.192	61.705	17.187	8.584	142.635	44.000	18.423
80	1.066	0.104	0.095	4.033	0.860	0.768	8.865	1.268	1.096	104.005	10.063	7.279	72.908	22.859	10.069	169.744	61.891	21.994
90	1.296	0.110	0.101	4.813	1.100	0.960	10.324	1.623	1.368	124.941	12.662	8.417	85.486	30.187	11.860	199.666	79.641	25.808
100	1.540	0.124	0.117	5.734	1.451	1.103	11.830	2.170	1.602	148.041	15.648	9.644	99.596	38.140	13.871	234.448	101.344	29.995

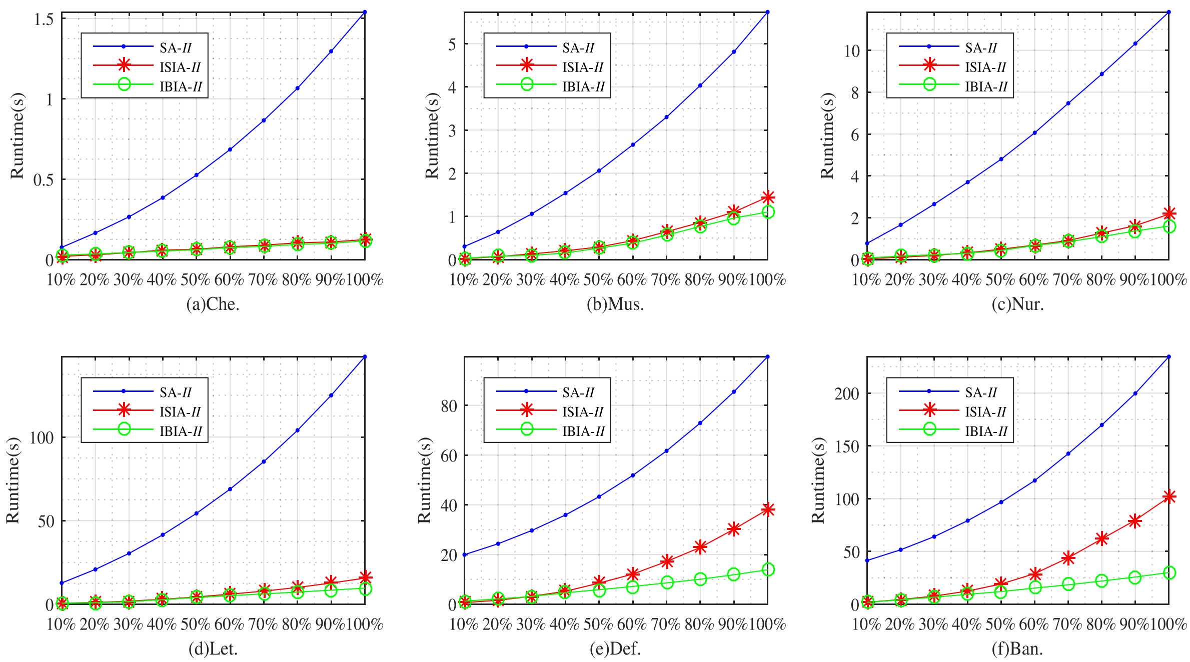

Fig. 3. Comparisons of runtime in static and incremental insertion algorithms of DqII-DTRS versus different inserting ratios.

the increase of deleting ratios, the computational time of SA-*I* decreases rapidly, while that of ISDA-*I* and IBDA-*I* increases slowly. Moreover, IBDA-*I* changes more smoothly and slowly than ISDA-*I*. So the proposed incremental deletion algorithms ISDA-*I* and IBDA-*I* are relatively stable. Therefore, ISDA-*I* is very efficient in dynamic maintenance of approximations of DqI-DTRS with the sequential deletion of objects when the deleting ratio is small and IBDA-*I* is very efficient in dynamic maintenance of approximations of DqI-DTRS in dynamic large data sets with the batch deletion of objects.

6.3. Comparisons of static and incremental algorithms of DqII-DTRS in decision systems with the variation of objects

In order to verify the computational efficiency of the incremental insertion and deletion algorithms of DqII-DTRS, we compare the incremental sequential and batch insertion algorithms of DqII-DTRS (ISIA-*II* and IBIA-*II*) with the static algorithm of DqII-DTRS (SA-*II*) in the case of inserting objects into data, and compare incremental sequential and batch deletion algorithms of DqII-DTRS (ISDA-*II* and IBDA-*II*) with the static algorithm of DqII-DTRS (SA-*II*) in the case of deleting objects from data, respectively. Detailed experimental results about the insertion case are shown in Table 10 and Fig. 3, and the experimental results about the deletion case are shown in Table 11 and Fig. 4.

From Table 10, in comparison with SA-*II*, both ISIA-*II* and IBIA-*II* greatly reduce the runtime of computing approximations for each data set.

First, we compare ISIA-*II* and SA-*II*. On data set Che., the runtime of ISIA-*II* shows a reduced by up to 8.05% of the time of SA-*II* at inserting ratio 100%. On the remaining five data sets, the runtime of ISIA-*II* is reduced by up to 6.58%, 5.51%, 4.93%, 3.37% and 4.56% of that of SA-*II* at inserting ratio 10%, respectively. In ten experiments of the six data sets, the runtime of ISIA-*II* is reduced to 13.64%, 16.30%, 11.09%, 7.97%, 21.12% and 23.61% on average of that of SA-*II*, respectively. So ISIA-*II* updates knowledge faster than SA-*II* in dynamic data sets with the sequential insertion of objects.

Then we compare IBIA-*II*, ISIA-*II* and SA-*II*. At each inserting ratio on each data set, the computational time of IBIA-*II* is always less than that of SA-*II*. On the six data sets, the runtime of IBIA-*II* is reduced by up to 7.60%, 9.25%, 8.22%, 4.51%, 6.28% and 4.33% of the time of SA-*II* at the corresponding inserting ratio 100%, 30%, 40%, 20%, 10%, 10%, respectively. In the ten experiments of the above six data sets, the number of times that IBIA-*II* performs better than ISIA-*II* is 7, 8, 7, 10, 8 and 10, respectively. On Che., Mus. and Nur., IBIA-*II* performs better than ISIA-*II* and on Let., Def. and Ban., the performance of IBIA-*II* is far better than that of ISIA-*II*. So in dynamic data sets with the batch insertion of objects, IBIA-*II* can help us to update the approximations of DqII-DTRS quickly.

Table 11
Comparisons of computational time about the static algorithm and incremental deletion algorithms of DqII-DTRS in deleting objects.

DR (%)	Che.			Mus.			Nur.			Let.			Def.			Ban.		
	SA-*II*	ISDA-*II*	IBDA-*II*	SA-*II*	ISDA-*II*	IBDA-*II*	SA-*II*	ISDA-*II*	IBDA-*II*	SA-*II*	ISDA-*II*	IBDA-*II*	SA-*II*	ISDA-*II*	IBDA-*II*	SA-*II*	ISDA-*II*	IBDA-*II*
5	1.384	0.027	0.025	5.150	0.332	0.108	10.920	0.114	0.111	131.233	1.020	0.705	87.505	5.281	1.967	238.041	16.016	3.047
10	1.160	0.056	0.040	4.334	0.519	0.194	9.520	0.235	0.201	110.528	1.859	1.125	74.880	11.140	3.455	208.151	30.157	5.202
15	0.950	0.079	0.053	3.600	0.688	0.284	8.163	0.356	0.307	90.866	2.486	1.504	63.037	16.172	4.498	181.791	44.813	6.849
20	0.770	0.100	0.075	2.960	0.865	0.365	6.850	0.469	0.431	73.522	3.047	1.855	52.896	21.093	5.254	158.698	58.156	8.300
25	0.613	0.119	0.094	2.360	1.008	0.432	5.570	0.583	0.533	58.770	3.541	2.183	44.146	24.828	5.970	132.885	69.516	9.640
30	0.470	0.136	0.113	1.820	1.125	0.502	4.424	0.692	0.628	45.789	3.983	2.478	36.755	28.843	6.646	109.338	79.078	10.880
35	0.349	0.146	0.131	1.340	1.215	0.566	3.340	0.790	0.719	34.624	4.374	2.763	30.302	31.968	7.248	88.713	87.391	12.001
40	0.240	0.171	0.149	0.930	1.305	0.626	2.385	0.890	0.818	25.011	4.764	3.002	24.709	34.906	7.818	71.885	94.672	13.011
45	0.150	0.185	0.172	0.558	1.385	0.676	1.470	0.990	0.894	16.793	5.020	3.223	20.255	37.062	8.337	56.807	101.672	13.886
50	0.070	0.205	0.195	0.250	1.448	0.721	0.690	1.090	1.004	10.019	5.283	3.403	16.709	39.093	8.791	44.667	108.781	14.700

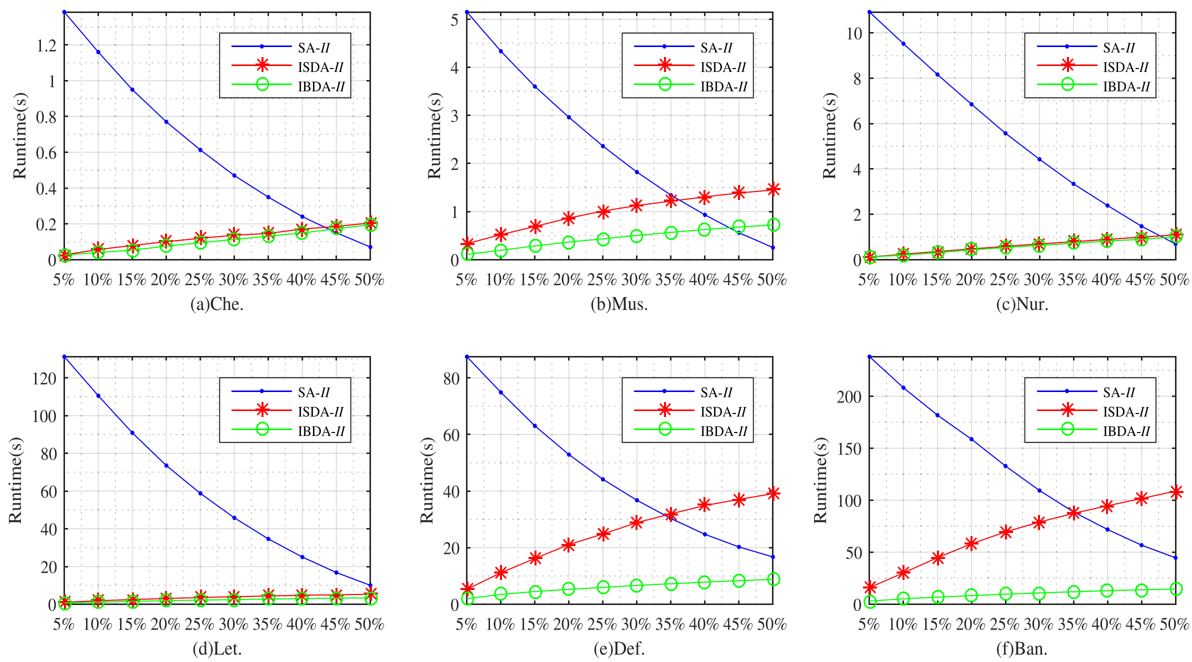

Fig. 4. Comparisons of runtime in static and incremental deletion algorithms of DqII-DTRS versus different deleting ratios.

Fig. 3 depicts intuitively the detailed changes of the runtime of SA-*II*, ISIA-*II* and IBIA-*II* with the increase of inserting ratios. It is obvious that at each experiment of each data set, the computation speed of ISIA-*II* and IBIA-*II* is faster than that of SA-*II*. Moreover, the computational time of SA-*II* increases sharply with the increase of inserting ratios for each data set, while that of ISIA-*II* and IBIA-*II* increases more slowly. Moreover, IBIA-*II* performs better than ISIA-*II* especially in large data sets.

Based on the analysis results of Table 10 and Fig. 3, we obtain that ISIA-*II* and IBIA-*II* are very efficient in dynamic maintenance of approximations for data sets with the sequential insertion of objects and the batch insertion of objects, respectively.

From Table 11, in comparison with SA-*II*, both ISDA-*II* and IBDA-*II* reduce the runtime of computing approximations to some extent for each data set.

First, we compare ISDA-*II* and SA-*II*. On six data sets, the runtime of ISDA-*II* is reduced by up to 1.95% , 6.45%, 1.04%, 0.78%, 6.04% and 6.73% of that of SA-*II* at deleting ratio 5%, respectively. On data set Let., ISDA-*II* always performs better than SA-*II* in ten experiments. On data sets Che. and Nur., only when the deleting ratio close to or equal to 50%, the performance of ISDA-*II* is slightly weaker than that of SA-*II*. On data set Mus., when the deleting ratio is not less than 40%, the runtime of ISDA-*II* is slightly larger than that of SA-*II*. On data sets Def. and Ban., when the deleting ratio is not less than 40%, SA-*II* performs better than ISDA-*II*. In ten experiments, the runtime of ISDA-*II* is reduced to 23.69%, 37.42%, 18.79%, 13.84%, 36.86% and 43.67% on average of that of SA-*II* on six data sets, respectively. So ISDA-*II* performs better than SA-*II* for calculating the approximations of DqII-DTRS in dynamic data sets with the sequential deletion of small objects (not more than 40% to 50% of the original data).

Then we compare IBDA-*II*, ISDA-*II* and SA-*II*. In the ten experiments of the above six data sets, the number of times that IBDA-*II* performs better than SA-*II* is 8, 8, 9, 10, 10 and 10, respectively. On six data sets, the runtime of ISDA-*II* is reduced to 19.95%, 22.78%, 17.07%, 8.80%, 20.49% and 11.90% on average of that of SA-*II*, respectively. In each experiment on each data set, IBDA-*III* performs better than ISDA-*II*. On six data sets, the runtime of IBDA-*II* is reduced by up to 67.09%, 32.53%, 85.53%, 60.50%, 22.40% and 13.51% of the time of ISDA-*II* at the corresponding deleting ratios 15%, 5%, 10%, 15%, 40% and 50%, respectively. So IBDA-*II* performs better than ISDA-*II* and SA-*II* for calculating the approximations of DqII-DTRS in dynamic large data sets with the batch deletion of objects.

Meanwhile, Fig. 4 depicts intuitively the changes of the runtime of SA-*II*, ISDA-*II* and IBDA-*II* with the increase of deleting ratios. We find that on each data set with the increase of deleting ratios, the runtime of ISDA-*II* and IBDA-*II* increases slowly, while that of SA-*II* decreases rapidly. So ISDA-*II* and IBDA-*II* are relatively stable. Moreover, for each data set, when the deleting ratio

Table 12
The average runtime under 25 pairs of parameters of incremental algorithms.

Runtime (s)	Che.	Mus.	Nur.	Let.	Def.	Ban.
ISIA-*I*	0.181 ± 0.013	1.753 ± 0.033	6.723 ± 0.236	18.440 ± 1.688	33.395 ± 2.766	87.081 ± 6.040
ISIA-*II*	0.133 ± 0.014	1.457 ± 0.034	2.186 ± 0.148	15.874 ± 1.497	38.253 ± 2.514	103.098 ± 6.476
ISDA-*I*	0.298 ± 0.015	1.711 ± 0.031	4.336 ± 0.179	36.408 ± 2.345	35.967 ± 2.964	93.828 ± 3.670
ISDA-*II*	0.214 ± 0.014	1.461 ± 0.023	1.106 ± 0.095	5.359 ± 0.785	38.989 ± 2.887	108.252 ± 6.608
IBIA-*I*	0.175 ± 0.012	1.471 ± 0.031	2.873 ± 0.166	9.425 ± 1.152	14.264 ± 1.170	31.300 ± 1.781
IBIA-*II*	0.126 ± 0.015	1.107 ± 0.027	1.601 ± 0.191	9.823 ± 1.114	14.103 ± 1.145	31.368 ± 2.073
IBDA-*I*	0.183 ± 0.015	0.973 ± 0.027	2.460 ± 0.234	3.725 ± 0.639	10.760 ± 1.072	15.416 ± 0.959
IBDA-*II*	0.204 ± 0.014	0.727 ± 0.030	1.106 ± 0.173	3.432 ± 0.612	9.087 ± 0.886	14.656 ± 1.596

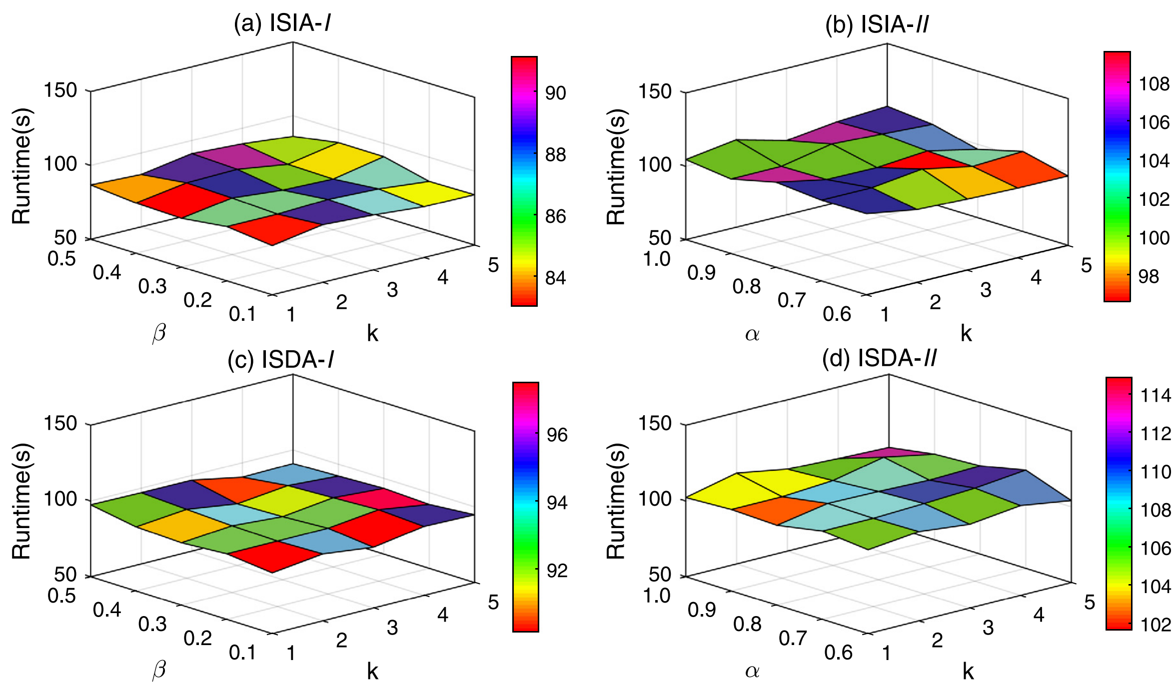

Fig. 5. The runtime of incremental sequential insertion and deletion algorithms of Dq-DTRS under different parameter values on Ban.

is less than or close to 40% or 50%, the runtime of ISDA-*II* is less than or approximately equal to that of SA-*II*. The computational time of IBDA-*II* is always less than that of ISDA-*II* and SA-*II* in dynamic large data sets.

Based on the analysis results of Table 11 and Fig. 4, we obtain that ISDA-*II* is very efficient in dynamic maintenance of approximations of DqII-DTRS with the sequential deletion of objects when the deleting ratio is small (less than or close to 40% or 50% of the original data) and IBDA-*II* is very efficient in dynamic maintenance of approximations of DqII-DTRS in dynamic large data sets with the batch deletion of objects.

6.4. The influence of parameters on the performances of incremental insertion and deletion algorithms of Dq-DTRS

In the last experiment of the above comparative experiments ($IR = 100\%$ and $DR = 50\%$), we set the value of β to vary from 0.1 to 0.5 in steps of 0.1, k to vary from 1 to 5 in steps of 1 and α to vary from 0.6 to 1.0 in steps of 0.1. The average runtime under 25 pairs of parameters of incremental insertion and deletion algorithms of DqI-DTRS and DqII-DTRS is shown in Table 12.

From Table 12, we can see that under 25 pairs of parameter values, the range of runtime of each incremental algorithms changes little in terms of median. For example, even on the large data Ban. with samples 45 211, for ISIA-*I*, there is a change of 6.040 s in terms of median 87.081 s; for ISIA-*II*, there is a change of 6.476 s in terms of median 103.098 s; for ISDA-*II*, there is a change of 6.608 s in terms of median 108.252 s. When these incremental algorithms (ISIA-*I*, ISIA-*II*, ISDA-*I*, ISDA-*II*, IBIA-*I*, IBIA-*II*) run in 30 s, their variations are about 2–3 s. Therefore, different values of parameters have little effect on the overall efficiency of the incremental insertion and deletion algorithms. This is consistent with Chen et al.'s research results [26,28] that when inserting objects into or deleting objects from data sets, the incremental algorithms has essentially the same runtime under different error tolerance parameters, with only a few small fluctuations.

Taking a large data set Ban. as an example, we give more intuitive results in Figs. 5–6. As can be seen from Figs. 5–6, the runtime of these incremental algorithms is basically in a contour plane under 25 pairs of parameter values. From Table 12 and Figs. 5–6, We can see that the proposed incremental sequential insertion, sequential deletion, batch insertion and batch deletion algorithms of Dq-DTRS is stable.

6.5. Experimental summary

By comparing static and incremental algorithms of DqI-DTRS and DqII-DTRS in Sections 6.2 and 6.3, we have obtained the following conclusions: (1) The proposed incremental sequential and batch insertion algorithms of Dq-DTRS models (ISIA-*I*, ISIA-*II*, IBIA-*I* and IBIA-*II*) are feasible and very efficient in dynamic maintenance of approximations for data sets with the sequential and batch insertion of objects. Especially when a large number of objects are inserted into data sets, these algorithms can help us

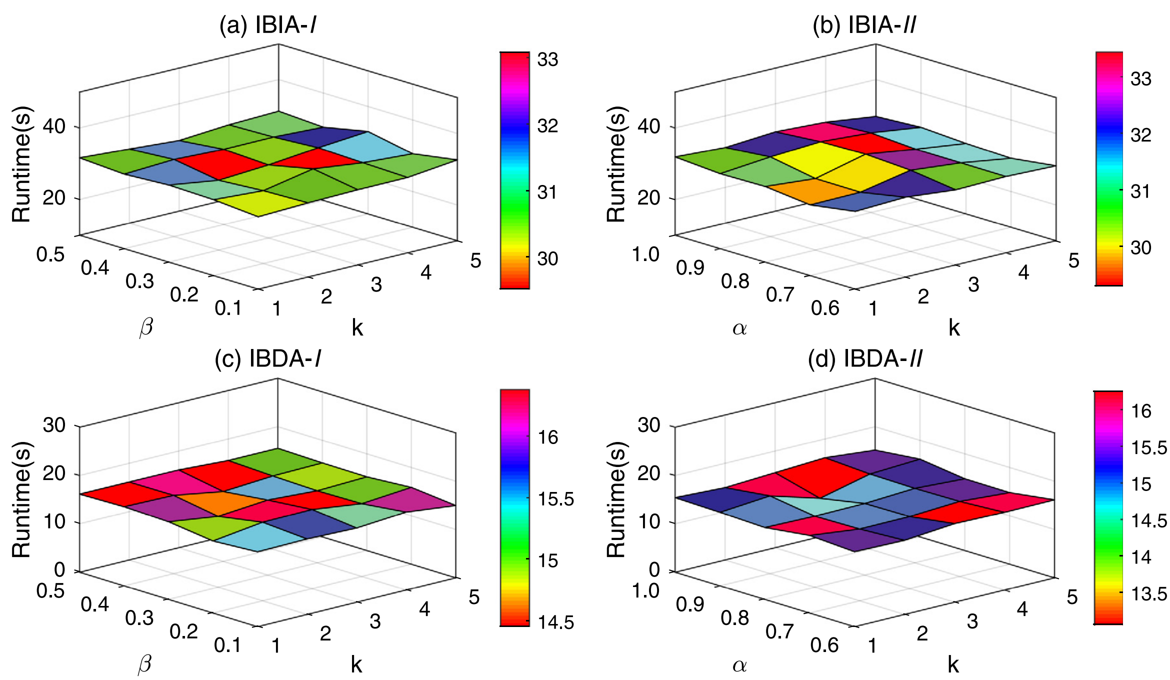

Fig. 6. The runtime of incremental batch insertion and deletion algorithms of Dq-DTRS under different parameter values on Ban.

update knowledge quickly. (2) The proposed incremental sequential and batch deletion algorithms of Dq-DTRS models (ISDA-*I*, ISDA-*II*, IBDA-*I* and IBDA-*II*) are feasible and efficient in dynamic maintenance of approximations for data sets with the sequential and batch deletion of objects. In particular, when relatively small objects are deleted from large data sets, these algorithms are very efficiency. (3) On small data sets such as Che. and Mus., incremental sequential insertion and deletion algorithms (ISIA-*I*, ISIA-*II*, ISDA-*I* and ISDA-*II*) have similar computational performance over the corresponding incremental batch algorithms in data sets with the batch variation of objects. (4) On large data sets such as Def. and Ban., incremental batch insertion and deletion algorithms (IBIA-*I*, IBIA-*II*, IBDA-*I* and IBDA-*II*) have absolute computational advantages over the corresponding incremental sequential algorithms in data sets with the batch variation of many objects.

7. Conclusion and future work

The development of information technology makes the scale of data larger and the real-time update speed faster. In real life, dynamic data of different types is universal. Making full use of the correlation of real time data and priori knowledge derived from previous data can more efficiently discover knowledge and rules from current data. In this paper, we first analyze systematically the variations of decision classes, equivalence classes, conditional probability, external grade and internal grade with the sequential and batch variations of objects. Then we propose incremental methods to update quickly approximations of Dq-DTRS in dynamic decision information systems with the variation of objects and design the corresponding incremental algorithms. The experimental comparisons show that our incremental approaches are feasible, stable and efficient in calculating approximation sets, which can help people express knowledge, extract rules and make sound decisions. Based on the above results, we will further study the incremental feature selection of the Dq-DTRS model and the dynamic approximation updating of Dq-DTRS in fuzzy dynamic data sets.

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